STATISTICAL INFERENCE FOR A COPULA-BASED SIMPLE STEP-STRESS ACCELERATED LIFE MODEL

by

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In this paper, we undertake a statistical analysis of a simple step-stress accelerated competing failure model. The prevailing hypothesis posits that the competing failure modes are mutually independent. However, empirical evidence suggests that failures tend to be dependent in practice. Consequently, we hypothesize that the failure mechanisms are interconnected. The dependence structure examined in this article is constructed by Gumbel-Hougard Copula and Clayton Copula, and different construction methods are compared. The estimation of unknown parameters is achieved through the maximum likelihood estimation method. The precision of these estimates is subsequently evaluated through Monte Carlo simulations. The simulation results demonstrate that: The Copula theory plays a monumental role in the study of the correlation of competitive failure mechanisms. In this regard, the Gumbel-Hougard Copula demonstrates superior performance in comparison to the Bivariate Clayton Copula within the same framework. Finally, the methods of inference discussed are illustrated with a real dataset.

Keywords: *bivariate Clayton Copula, competing risk, Gumbel-Hougard Copula, step-stress accelerated life test.*

Introduction

Accelerated life testing (ALT) is a method of acquiring failure data under standard usage conditions during life testing. The objective of testing is to swiftly gather data that, when accurately modeled and evaluated, can yield the sought-after information regarding the product's life or performance under standard usage conditions. Nelson [1] elucidated that, in ALT, test units are subjected to elevated levels of stress, thereby inducing rapid breakdowns. As demonstrated in the following references, stress can be applied in a variety of ways [2-5]. A particularly noteworthy approach is the step-stress accelerated life testing (S-SALT) method, a specialized ALT method that enables researchers to incrementally increase stress levels at predetermined intervals during testing. This feature ensures maximum adaptability and flexibility, making it a valuable tool in the field. A simple S-SALT is defined as a particular S-SALT in which only two stress levels are studied. A model that correlates the life-spans of failure under varying stress levels, such as the cumulative exposure model (CEM) or the tempered failure rate model, is indispensable for the analysis of S-SALT modeling data. The

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CEM, which was pioneered by Sedyakin [6], is regarded by many as the most prominent model in this field. A substantial corpus of literature has meticulously examined the S-SALT under the CEM assumption. For illustrative purposes, please refer to [7-12]. Given the intricate nature of the structure, it is improbable that product failure is attributable to a solitary risk factor. Consequently, the competing failure model is a general model in research on reliability, as evidenced in [13-15].

In a multitude of statistical models of S-SALT with competing failures, the experimenter frequently posits the hypothesis that the competing failure modes are mutually independent. The construction of the joint distribution of the risk variables is a formidable task. However, empirical evidence frequently reveals a tendency for these failures to be dependent in practice. Copula, a sophisticated mathematical apparatus, effectively eliminates this restriction. The restriction that the joint distribution is formed from the same family of marginal distributions is abolished by Copula. The extant literature contains several notable Copula models, as outlined by Nelsen [16]. However, there is a paucity of literature addressing the application of Copula functions to S-SALT, as evidenced by the works of Ghaly *et al.* [17] and Kotb and El-Din [18]. The extant literature employs a single Copula function, precluding a comprehensive comparison of different Copula functions. In contrast, the present article proposes a novel approach by employing the Clayton Copula and the Gumbel-Hougard Copula (GHC) to model dependence.

Assumptions and model description

Assumptions and model description

The following assumptions are posited to elucidate the S-SALT model:

– Each test unit is configured with two competing failure cases. The interdependence between these two failure modes is modeled using the GHC and the Bivariate Clayton Copula (BCC):

$$C_{\mathrm{GH}}(u,v)=\exp\left\{-[(-\ln u)^{\theta}+(-\ln v)^{\theta}]^{1/\theta}\right\} \tag{1}$$

$$C_C(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta} \tag{2}$$

and the most renowned measure is Kendall's tau, then $\tau_{\mathrm{GH}}(\theta-1)/\theta$ is of GHC, and $\tau_{\mathrm{C}}\ \theta/(\theta+2)$ is of BCC. The expiration time is the minimum of its two potential lifetimes.

– A test unit life follows an exponential distribution, and its scale parameter is λ_{ij}: .

$$F_{ij}(t)=1-\exp(-\lambda_{ij}t) \tag{3}$$

$$f_{ij}(t)=\lambda_{ij}\exp(-\lambda_{ij}t) \tag{4}$$

where $t\geq 0$, $\lambda_{ij}>0$, and $i, j = 1.2$.

– The scale parameter λ_{ij} is a log-linear function of the stress level S_i:

$$\ln\lambda_{ij}=a_j+b_j\varphi(S_i) \tag{5}$$

Herein the Arrhenius model is selected, and $\varphi(S_i)=1/S_i$, a_i, and b_j are uncertain parameters.

– It is assumed that a CEM is associated with the distribution of lifetime exposure to varying levels of stress:

$$F_1(t_1)=F_2(t_2) \tag{6}$$

– The simple S-SALT data are progressively Type-II censored (PT-IIC).

In PT-IIC scheme, the simple S-SALT is sketched as: At the starting level S_1, a random selection of n test units are put through the experiment. The failure-times are noted along with the failure cause index $\delta_i(\delta_i \in \{1, 2\})$. When the pre-fixed number of failure N_1 is reached under S_1, then the level is swelled, and the test goes on until the failure is seen at the level S_2. At the i^{th} ($i = 1, 2,\ldots, N_1 + N_2$) failure time, R_i ($i = 1,\ldots, N_1 + N_2$) of the surviving units are indiscriminately taken away, where N_1, N_2, $R_1,\ldots, R_{N_1+N_2}(N_1 + N_2 + R_1 +\ldots + R_{N_1+N_2} = n)$ are prefixed constants. Therefore, we have the observed data as:

$$S_1 : (t_{1:n}, \delta_1, R_1), (t_{2:n}, \delta_2, R_2), \cdots, (t_{N_1:n}, \delta_{N_1}, R_{N_1})$$

$$S_2 : (t_{N_1+1:n}, \delta_{N_1+1}, R_{N_1+1}), (t_{N_1+2:n}, \delta_{N_1+2}, R_{N_1+2}), \cdots, (t_{N_1+N_2:n}, \delta_{N_1+N_2}, R_{N_1+N_2})$$

where $t_{1:n}, \cdots, t_{N_1+N_2:n}$ are order statistics.

Equations (7) and (8) display the CDF and the corresponding PDF of the lifetime for the j^{th} ($j = 1, 2$) failure cause, respectively.

$$F_j(t:\lambda_{1j}, \lambda_{2j}) = \begin{cases} F_{1j}(t) = 1 - \exp(-\lambda_{1j} t) & (0 \le t \le \tau) \\ F_{2j}(t) = 1 - \exp[-(\lambda_{1j} - \lambda_{2j})\tau - \lambda_{2j} t] & (t > \tau) \end{cases} \tag{7}$$

$$f_j(t:\lambda_{1j}, \lambda_{2j}) = \begin{cases} f_{1j}(t) = \lambda_{1j} \exp(-\lambda_{1j} t) & (0 \le t \le \tau) \\ f_{2j}(t) = \lambda_{2j} \exp[-(\lambda_{1j} - \lambda_{2j})\tau - \lambda_{2j} t] & (t > \tau) \end{cases} \tag{8}$$

where $\tau \in (t_{N_1:n}, t_{N_1+1:n})$..

Likelihood function

Based on the first failure mode, the CDF and the corresponding PDF of the lifetime of the product under stress level S_i can be obtained:

$$F^{(i1)}(t) = P(T_{i2} > T_{i1}, \quad T_{i1} \le t) \tag{9}$$

$$f^{(i1)}(t) = \frac{\mathrm{d}F^{(i1)}(t)}{\mathrm{d}t} = f_{i1}(t) \frac{\partial C(u,v)}{\partial u}\Bigg|_{\substack{u=S_{i1}(t) \\ v=S_{i2}(t)}} \tag{10}$$

where $u = S_{i1}(t) = 1 - F_{i1}(t)$ and $v = S_{i2}(t) = 1 - F_{i2}(t)$.

When:

$$C_{GH}(u,v) = \exp\left\{-\left[(-\ln u)^\theta + (-\ln v)^\theta\right]^{1/\theta}\right\}$$

we can obtain the following result by substituting eqs. (1), (7) and (8) into eq. (10).

$$f_{GH}^{11}(t_{i:n}) = \lambda_{11}^\theta \times (\lambda_{11}^\theta + \lambda_{12}^\theta)^{(1-\theta)/\theta} \times \exp\left[-(\lambda_{11}^\theta + \lambda_{12}^\theta)^{1/\theta} t_{i:n}\right] \tag{11a}$$

$$f_{GH}^{(21)}(t_{i:n}) = \lambda_{21} \times \left[\left(\lambda_{11} - \lambda_{21} \right) \tau + \lambda_{21} t_{i:n} \right]^{\theta\text{-}1} \times$$

$$\times \left\{ \left[\left(\lambda_{11} - \lambda_{21} \right) \tau + \lambda_{21} t_{i:n} \right]^{\theta} + \left[\left(\lambda_{12} - \lambda_{22} \right) \tau + \lambda_{22} t_{i:n} \right]^{\theta} \right\}^{(1-\theta)/\theta} \times$$

$$\times \exp\left(- \left\{ \left[\left(\lambda_{11} - \lambda_{21} \right) \tau + \lambda_{21} t_{i:n} \right]^{\theta} + \left[\left(\lambda_{12} - \lambda_{22} \right) \tau + \lambda_{22} t_{i:n} \right]^{\theta} \right\}^{1/\theta} \right) \tag{11b}$$

Based on the second failure mode, the CDF and the corresponding PDF of the lifetime of the product under stress level S_i can be obtained:

$$F^{(i2)}(t) = P(T_{i1} > T_{i2}, \quad T_{i2} \le t) \tag{12}$$

$$f^{(i2)}(t) = \frac{\mathrm{d}F^{(i2)}(t)}{\mathrm{d}t} = f_{i2}(t) \times \frac{\partial C(u,v)}{\partial v} \bigg|_{\substack{u=S_{i1}(t) \\ v=S_{i2}(t)}} \tag{13}$$

By substituting eqs. (1), (7) and (8) into eq. (13) can result in the following result:

$$f_{GH}^{12}(t_{i:n}) = \lambda_{12}^{\theta} \times (\lambda_{11}^{\theta} + \lambda_{12}^{\theta})^{(1\text{-}\theta)/\theta} \times \exp\left[-(\lambda_{11}^{\theta} + \lambda_{12}^{\theta})^{1/\theta} t_{i:n} \right] \tag{14 a}$$

$$f_{GH}^{(22)}(t_{i:n}) = \lambda_{22} \times \left[\left(\lambda_{12} - \lambda_{22} \right) \tau + \lambda_{22} t_{i:n} \right]^{\theta\text{-}1} \times$$

$$\times \left\{ \left[\left(\lambda_{11} - \lambda_{21} \right) \tau + \lambda_{21} t_{i:n} \right]^{\theta} + \left[\left(\lambda_{12} - \lambda_{22} \right) \tau + \lambda_{22} t_{i:n} \right]^{\theta} \right\}^{(1-\theta)/\theta} \times$$

$$\times \exp\left(- \left\{ \left[\left(\lambda_{11} - \lambda_{21} \right) \tau + \lambda_{21} t_{i:n} \right]^{\theta} + \left[\left(\lambda_{12} - \lambda_{22} \right) \tau + \lambda_{22} t_{i:n} \right]^{\theta} \right\}^{1/\theta} \right) \tag{14b}$$

According to assumptions (1) and (2), the SF under stress S_i can be seen in eq. (15):

$$S_{GH1}(t) = \exp\left[-\left(\lambda_{11}^{\theta} + \lambda_{12}^{\theta} \right)^{1/\theta} t \right] \tag{15a}$$

$$S_{GH2}(t) = \exp\left(- \left\{ \left[\left(\lambda_{11} - \lambda_{21} \right) \tau + \lambda_{21} t \right]^{\theta} + \left[\left(\lambda_{12} - \lambda_{22} \right) \tau + \lambda_{22} t \right]^{\theta} \right\}^{1/\theta} \right) \tag{15b}$$

Then, the likelihood function of the failure sample is:

$$L\left(t \middle| \lambda_{ij}, \theta \right) \propto \prod_{i=1}^{N_1} \left\{ \left[f^{(11)}(t_{i:n}) \right]^{I_1(\delta_i)} \left[f^{(12)}(t_{i:n}) \right]^{I_2(\delta_i)} S_1\left(t_{i:n} \right)^{R_i} \right\} \times$$

$$\times \prod_{i=N_1+1}^{N_1+N_2} \left\{ \left[f^{(21)}(t_{i:n}) \right]^{I_1(\delta_i)} \left[f^{(22)}(t_{i:n}) \right]^{I_2(\delta_i)} S_2\left(t_{i:n} \right)^{R_i} \right\} \tag{16}$$

Let $\mathbf{\Theta} = (\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \theta)$, by substituting eqs. (11), (14) and (15) into eq. (16), we can obtain the following equation:

$$L_{GH}\left(t|\mathbf{\Theta}\right) \propto \lambda_{11}^{\theta n_{11}} \times \lambda_{12}^{\theta n_{12}} \times \lambda_{21}^{n_{21}} \times \lambda_{22}^{n_{22}} \left(\lambda_{11}^{\theta}+\lambda_{12}^{\theta}\right)^{(1-\theta)\cdot N_1/\theta} \times \exp\left\{-\left(\lambda_{11}^{\theta}+\lambda_{12}^{\theta}\right)^{1/\theta} T_1\right\}\times$$

$$\times\prod_{i=N_1+1}^{N_1+N_2}\left\{\left[\left(\lambda_{11}-\lambda_{21}\right)\tau+\lambda_{21}t_{i:n}\right]^{(\theta-1)I_1(\delta_i)} \times\left[\left(\lambda_{12}-\lambda_{22}\right)\tau+\lambda_{22}t_{i:n}\right]^{(\theta-1)I_2(\delta_i)}\right\}\times$$

$$\times\prod_{i=N_1+1}^{N_1+N_2}\left\{\left[\left(\lambda_{11}-\lambda_{21}\right)\tau+\lambda_{21}t_{i:n}\right]^{\theta}+\left[\left(\lambda_{12}-\lambda_{22}\right)\tau+\lambda_{22}t_{i:n}\right]^{\theta}\right\}^{(1-\theta)/\theta}\times$$

$$\times\exp\left\{-\left\{\sum_{i=N_1+1}^{N_1+N_2}\left\{\left[\left(\lambda_{11}-\lambda_{21}\right)\tau+\lambda_{21}t_{i:n}\right]^{\theta}+\left[\left(\lambda_{12}-\lambda_{22}\right)\tau+\lambda_{22}t_{i:n}\right]^{\theta}\right\}^{1/\theta}\left(1+R_i\right)\right\}\tag{17}$$

where

$$n_{11}=\sum_{i=1}^{N_1}I_1(\delta_i),\quad n_{12}=\sum_{i=1}^{N_1}I_2(\delta_i),\quad n_{21}=\sum_{i=N_1+1}^{N_1+N_2}I_1(\delta_i)$$

$$n_{22}=\sum_{i=N_1+1}^{N_1+N_2}I_2(\delta_i),\quad T_1=\sum_{i=1}^{N_1}(1+R_i)t_{i:n}$$

When $C_C(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$ we can obtain the likelihood function of the failure sample in the same way, which is:

$$L_C\left(t|\mathbf{\Theta}\right)\propto\lambda_{11}^{n_{11}}\times\lambda_{12}^{n_{12}}\times\lambda_{21}^{n_{21}}\times\lambda_{22}^{n_{22}}\times\exp\left(\theta\lambda_{11}T_{11}\right)\times\exp\left(\theta\lambda_{12}T_{12}\right)\times$$

$$\times\exp\left(\lambda_{21}\theta T_{21}\right)\times\exp\left(\lambda_{22}\theta T_{22}\right)\times\prod_{i=1}^{N_1}\left[\exp(\theta\lambda_{11}t_{i:n})+\exp(\theta\lambda_{12}t_{i:n})-1\right]^{-(1+\theta+R_i)/\theta}\times$$

$$\times\prod_{i=N_1+1}^{N_1+N_2}\left\{\exp\left[\left(\lambda_{11}-\lambda_{21}\right)\tau\theta+\lambda_{21}\theta t_{i:n}\right]+\exp\left[\left(\lambda_{12}-\lambda_{22}\right)\tau\theta+\lambda_{22}\theta t_{i:n}\right]-1\right\}^{-(1+\theta+R_i)/\theta}\tag{18}$$

where

$$n_{11}=\sum_{i=1}^{N_1}I_1(\delta_i),\quad n_{12}=\sum_{i=1}^{N_1}I_2(\delta_i),\quad n_{21}=\sum_{i=N_1+1}^{N_1+N_2}I_1(\delta_i),\quad n_{22}=\sum_{i=N_1+1}^{N_1+N_2}I_2(\delta_i)$$

$$T_{11}=\sum_{i=1}^{N_1}t_{i:n}I_1(\delta_i)+\tau n_{21},\quad T_{12}=\sum_{i=1}^{N_1}t_{i:n}I_2(\delta_i)+\tau n_{22}$$

$$T_{21}=\sum_{i=N_1+1}^{N_1+N_2}t_{i:n}I_1(\delta_i)-\tau n_{21},\quad T_{22}=\sum_{i=N_1+1}^{N_1+N_2}t_{i:n}I_2(\delta_i)-\tau n_{22}$$

Maximum likelihood estimates

Taking the natural logarithm of the eq. (17), we have:

$$l_{GH}(t|\Theta)=\theta n_{11}\ln\lambda_{11}+\theta n_{12}\ln\lambda_{12}+n_{21}\ln\lambda_{21}+n_{22}\ln\lambda_{22}+\frac{1-\theta}{\theta}N_1\ln\left(\lambda_{11}{}^{\theta}+\lambda_{12}{}^{\theta}\right)-$$
$$-\left(\lambda_{11}{}^{\theta}+\lambda_{12}{}^{\theta}\right)^{1/\theta}T_1-\sum_{i=N_1+1}^{N_1+N_2}\left(\left\{\left[\left(\lambda_{11}-\lambda_{21}\right)\tau+\lambda_{21}t_{i:n}\right]^{\theta}+\left[\left(\lambda_{12}-\lambda_{22}\right)\tau+\lambda_{22}t_{i:n}\right]^{\theta}\right\}^{1/\theta}(1+R_i)\right)+$$
$$+\frac{1-\theta}{\theta}\sum_{i=N_1+1}^{N_1+N_2}\ln\left\{\left[\left(\lambda_{11}-\lambda_{21}\right)\tau+\lambda_{21}t_{i:n}\right]^{\theta}+\left[\left(\lambda_{12}-\lambda_{22}\right)\tau+\lambda_{22}t_{i:n}\right]^{\theta}\right\}+$$
$$+\left(\theta-1\right)\sum_{i=N_1+1}^{N_1+N_2}\left\{I_1(\delta_i)\ln\left[(\lambda_{11}-\lambda_{21})\tau+\lambda_{21}t_{i:n}\right]I_2(\delta_i)\ln\left[(\lambda_{12}-\lambda_{22})\tau+\lambda_{22}t_{i:n}\right]\right\} \tag{19}$$

Taking the natural logarithm of the eq. (18), we obtain:

$$l_C(t|\mathbf{\Theta})=n_{11}\ln\lambda_{11}+n_{12}\ln\lambda_{12}+n_{21}\ln\lambda_{21}+n_{22}\ln\lambda_{22}+\theta\lambda_{11}T_{11}+\theta\lambda_{12}T_{12}+\theta\lambda_{21}T_{21}+\theta\lambda_{22}T_{22}-$$
$$-\sum_{i=1}^{N_1}\frac{1+\theta+R_i}{\theta}\ln\left[\exp(\theta\lambda_{11}t_{i:n})+\exp(\theta\lambda_{12}t_{i:n})-1\right]-$$
$$-\sum_{i=N_1+1}^{N_1+N_2}\frac{1+\theta+R_i}{\theta}\ln\left\{\exp\left[(\lambda_{11}-\lambda_{21})\tau\theta+\lambda_{21}\theta t_{i:n}\right]+\exp\left[(\lambda_{12}-\lambda_{22})\tau\theta+\lambda_{22}\theta t_{i:n}\right]-1\right\} \tag{20}$$

We can take the first partial derivative of eq. (19) or eq. (20) with respect to λ_{ij} and θ. However these equations are non-linear, and the maximum likelihood estimates (MLE) of the parameter λ_{ij} and θ cannot be solved analytically with ease. So an iterative method like the homotopy perturbation method [19], the variational iteration method [20], the point solution technology [21], and modified Babylonian Algorithm [22] has to be used for numerical analysis.

Putting $\hat{\lambda}_{i1}$ and $\hat{\lambda}_{i2}$ into eq. (5), we can obtain the $\hat{a}_j$ and $\hat{b}_j$, and the parameters at the normal stress level are obtained:

$$\hat{a}_j=\frac{\ln\hat{\lambda}_{1j}\varphi(S_2)-\ln\hat{\lambda}_{2j}\varphi(S_1)}{\varphi(S_2)-\varphi(S_1)},\quad\text{and}\quad \hat{b}_j=\frac{\ln\hat{\lambda}_{2j}-\ln\hat{\lambda}_{1j}}{\varphi(S_2)-\varphi(S_1)} \tag{21}$$

So the parameters at the normal stress level $\hat{\lambda}_{0j}$ $(j=1,2)$ can be obtained:

$$\hat{\lambda}_{0\mathrm{j}}=\exp\left[\hat{a}_j+\hat{b}_j\varphi(S_0)\right] \tag{22}$$

Simulation study

Data generation and Simulation design

In this section, we use Monte Carlo simulations to compare different methods for different sample size and different progressive censoring schemes which are shown in tab. 1. Let us assume the normal stress level is S_0 = 293 K, S_1 = 323 K, S_2 = 353 K. The scale parameters are λ_{11} = 1.0, λ_{12} = 0.5, λ_{21} = 2.0, and λ_{22} = 1.0. We let the parameter of GHC and BCC reflect the τ = 1/3, τ = 1/2, and τ = 3/5. Thus, the copula parameter for BCC is θ = 1, 2, 3, and for GHC is θ = 1.5, 2, 2.5. By employing 1000 simulations to prevent randomness, we can acquire the average estimates (AE) and the mean square errors (MSE) of the MLE. Tables 2 and 3 present the AE and MSE of the λ_{ij} under different τ. In tabs. 4 and 5, the estimates of the $\hat{\lambda}_{0j}\,(j=1,2)$ are shown according to eq. (21). Furthermore, the CDF and true CDF for BCC and GHC under the same scheme are shown in fig. 1.

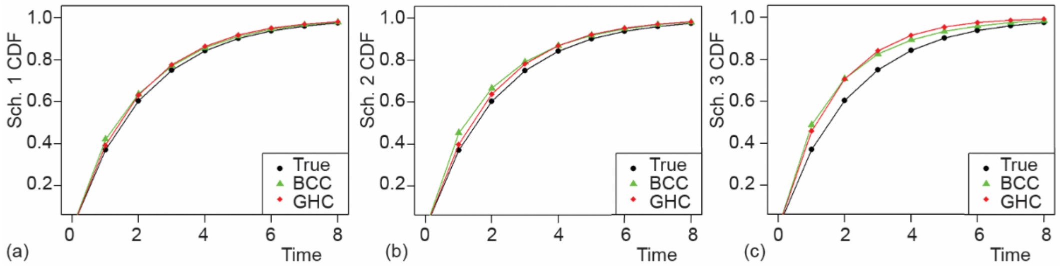

Figure 1. The CDF and true CDF for BCC and GHC under the same scheme

Table 1. The prefixed sample sizes

Scheme	N	N_1	N_2	$\sum_{i=1}^{N_1} R_i$	$\sum_{i=N_1+1}^{N_2} R_i$	$(R_1,\cdots,R_{N_1})(R_{N_1+1},\cdots,R_{N_1+N_2})$
1	40	20	10	5	5	(0, …, 0, 2, 0, 3)(0, …, 0, 3, 0, 2)
2	40	10	20	5	5	(0, …, 0, 2, 0, 3)(0, …, 0, 3, 0, 2)
3	60	30	16	8	6	(0, …, 0, 3, 1, 3, 1)(0, …, 0, 3, 1, 3, 1)
4	60	16	30	6	8	(0, …, 0, 0, 3, 3)(0, …, 0, 2, 0, 3, 3)

Table 2. The AE and MSE of the λ_{ij} under different τ for GHC

	Sch.	$\hat{\lambda}_{11\text{MLE}}^{GH}$		$\hat{\lambda}_{12\text{MLE}}^{GH}$		$\hat{\lambda}_{21\text{MLE}}^{GH}$		$\hat{\lambda}_{22\text{MLE}}^{GH}$	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE
τ = 1/3	1	1.109	0.118	0.527	0.101	2.145	0.230	0.930	0.251
	2	1.107	0.114	0.525	0.099	2.133	0.227	0.937	0.246
	3	1.098	0.109	0.520	0.094	2.109	0.226	0.944	0.239
	4	1.090	0.108	0.513	0.090	2.105	0.221	0.947	0.233
τ = 1/2	1	1.101	0.115	0.520	0.102	2.134	0.237	0.940	0.243
	2	1.098	0.113	0.513	0.101	2.121	0.230	0.953	0.239
	3	1.073	0.107	0.473	0.097	2.105	0.224	1.056	0.231
	4	1.068	0.103	0.483	0.093	2.099	0.219	1.074	0.227
τ = 3/5	1	1.099	0.111	0.475	0.099	2.128	0.227	1.052	0.232
	2	1.089	0.103	0.479	0.097	2.121	0.221	1.048	0.225
	3	1.068	0.101	0.482	0.089	2.071	0.213	1.043	0.211
	4	1.066	0.095	0.485	0.088	2.070	0.209	1.038	0.204

Table 3. The AE and MSE of the λ_{ij} under different τ for BCC

	Sch.	$\hat{\lambda}^{GH}_{11\text{MLE}}$		$\hat{\lambda}^{GH}_{12\text{MLE}}$		$\hat{\lambda}^{GH}_{21\text{MLE}}$		$\hat{\lambda}^{GH}_{22\text{MLE}}$	
		AE	MSE	AE	MSE	AE	MSE	AE	MSE
τ = 1/3	1	1.112	0.125	0.529	0.105	2.115	0.227	0.927	0.245
	2	1.095	0.120	0.517	0.107	2.106	0.225	0.934	0.230
	3	1.101	0.117	0.524	0.097	2.112	0.210	0.933	0.240
	4	1.105	0.121	0.478	0.099	2.119	0.211	0.953	0.189
τ = 1/2	1	1.091	0.120	0.525	0.097	2.104	0.214	0.930	0.237
	2	1.081	0.125	0.472	0.092	2.092	0.221	0.938	0.227
	3	1.087	0.115	0.520	0.085	2.097	0.204	0.937	0.224
	4	1.092	0.121	0.480	0.090	2.116	0.207	0.950	0.207
τ = 3/5	1	1.082	0.117	0.483	0.093	2.091	0.207	0.936	0.224
	2	1.091	0.121	0.480	0.097	2.083	0.211	0.937	0.223
	3	1.076	0.116	0.478	0.087	2.089	0.196	0.942	0.216
	4	1.089	0.123	0.485	0.090	2.110	0.202	0.956	0.201

Table 4. The scales parameters at the normal stress level for GHC

Sch.	$\hat{\lambda}^{GH}_{01}$			$\hat{\lambda}^{GH}_{02}$		
	τ = 1/3	τ = 1/2	τ = 3/5	τ = 1/3	τ = 1/2	τ = 3/5
1	0.5009	0.4960	0.4957	0.2658	0.2548	0.1822
2	0.5023	0.4967	0.4878	0.2613	0.2433	0.1865
3	0.5001	0.4764	0.4809	0.2535	0.1797	0.1902
4	0.4933	0.4732	0.4792	0.2451	0.1844	0.1939

Table 5. The scales parameters at the normal stress level for BCC

Sch.	$\hat{\lambda}^{C}_{01}$			$\hat{\lambda}^{C}_{02}$		
	τ = 1/3	τ = 1/2	τ = 3/5	τ = 1/3	τ = 1/2	τ = 3/5
1	0.5125	0.4945	0.4892	0.2691	0.2636	0.2049
2	0.4979	0.4879	0.5005	0.2535	0.2563	0.2144
3	0.5023	0.4925	0.4838	0.2637	0.2557	0.2111
4	0.5042	0.4921	0.4908	0.2081	0.2108	0.2141

Results of simulation study

From the simulations outcomes, we can draw these conclusions.

– Observing tabs. 2 and 3, we note that the estimation of $\hat{\lambda}_{ij}$ are closed to the real values and the MSE of parameters become less as N increasing under τ = 1/3, τ = 1/2, and τ = 3/5, respectively.

– Observing tabs. 2 and 3, we note that the estimation of $\hat{\lambda}_{ij}$ are closed to the real values, and the MSE of parameters diminish as τ increasing, implying that the correlation coefficient between failure mechanisms will have an effect on the accuracy of the parameters estimation.
– From fig. 1, the CDF in case of using GHC is closer to the true CDF than in case of using BCC under the same scheme.

A real life data

In this section, the proposed model from Han and Kundu [23] is illustrated using real-life data. Two failure modes have been identified. The two failure modes, Capacitor Failure and Controller Failure, are denoted as Model 1 and Model 2, respectively. The original data set is listed in tab 6. Employing the aforementioned approach results in the estimation of the unknown parameters, as illustrated in tab. 7.

Table 6. The real life data

Temperature	Failure times and failure causes (1 = capacitor and 2 = controller)
293 K	0.140(1), 0.738(2), 1.324(2), 1.582(1), 1.716(2), 1.794(2), 1.883(2), 2.293(2),2.660(2), 2.674(2), 2.725(2), 3.085(2), 3.924(2), 4.396(2), 4.612(1),4.892(2)
353 K	5.002(1), 5.022(2), 5.082(2), 5.112(1), 5.147(1), 5.238(1), 5.244(1), 5.247(1),5.305(1), 5.337(2), 5.407(1), 5.408(2), 5.445(1), 5.483(1), 5.717(2)

Table 7. The estimation of parameter for the real life data

Par.	λ_{11}	λ_{12}	λ_{21}	λ_{22}	$\hat{a}_1$	b_1	$\hat{a}_2$	b_2
BCC	0.021	0.094	1.107	0.588	19.5	–6834	8.42	–3160
GHC	0.023	0.088	1.350	0.655	20.2	–7020	9.38	–3460

Conclusion

In contrast to conventional statistical analysis [24, 25], the present study, within the framework of the PT-IIC scheme, undertakes a meticulous examination of the statistical analysis and reliability estimation of simple S-SALT. The correlation of failure mechanism is described by GHC function and BCC function. The ensuing presentation of data simulation results merits particular attention: the significance of Copula theory in examining the correlation between competitive failure mechanisms and GHC performance exceeding BCC in the same scheme. The methods of inference discussed are illustrated with a real dataset. For future research endeavors, the development of a methodology for selecting the most suitable copula models is imperative.

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