UNLEASHING THE POTENTIAL
Idempotent and (*k*+1)-Potent Matrices in MEMS
A Comprehensive Note on Linear Combinations

by

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In micro-electro-mechanical systems (MEMS), noise interference poses significant challenges to the reliability and performance of sensors. This study explores the role of matrix analysis in addressing these challenges, focusing on linear combinations of idempotent and (k+1)-potent matrices. The proposed methodology involves the introduction of a matrix $T = \alpha A + \beta B$*, where A is idempotent, B is (k+1)-potent, and* (α, β) *are non-zero complex numbers. The central aim of this study is to derive the necessary and sufficient conditions for T to be involutive, a property that is critical for the successful removal of noise in MEMS applications. Through a rigorous theoretical analysis, we establish these conditions and present them as actionable criteria, supported by lemmas and theorems. The results of this study contribute to a more profound comprehension of matrix interactions and offer valuable insights for the enhancement of MEMS design. This work establishes a theoretical framework integrating matrix algebra with applied engineering, thereby paving the way for the development of enhanced noise mitigation strategies in the field of MEMS technology.*

Key words: *MEMS, idempotent matrix, (k+1)-potent matrix, involutive matrix, linear combination*

Introduction

In the continuously evolving landscape of MEMS, as referenced in [1, 2], engineers and researchers are ceaselessly in pursuit of innovative solutions to surmount a multitude of challenges. One domain that has emerged as vitally important in the design and analysis of MEMS is the study of matrices, particularly for the removal of multiple types of noise during various advanced applications, as noted in [3, 4]. When MEMS systems are employed as sensors, diverse noises can significantly impact their reliability and accuracy. It has been discovered that idempotent matrices can be utilized more effectively for the removal of multiple noises, as indicated in [5-7].

Among the different types of matrices, idempotent matrices and $(k + 1)$-potent matrices have gained particular attention. An idempotent matrix, A, of order n has the property that when multiplied by itself, it remains unchanged. Similarly, a $(k + 1)$-potent matrix, B, has the characteristic that when raised to the power of $(k + 1)$, it becomes equal to itself.

In many MEMS applications, the relationship between these two types of matrices is of great significance [8]. However, it is often observed that the product of an idempotent ma-

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trix A and a (k + 1)-potent matrix B, denoted as AB, is not equal to BA. This asymmetry in matrix multiplication can have profound implications for the behavior and performance of MEMS devices.

To better understand and address these issues, we introduce a new matrix, T, defined as c_1A + c_2B, where c_1 and c_2 are non-zero complex numbers. By analyzing the properties of matrix, T, we aim to determine the necessary and sufficient conditions for T to be involutive. In this paper, through a comprehensive analysis of idempotent and (k + 1)-potent matrices and their combinations, we hope to provide valuable insights and solutions for MEMS design and analysis.

Preliminaries

Let A be a square matrix. The A is said to be k-potent if $A^k = A$. The A is called an idempotent matrix if $A^2 = A$. We say that A is involutive if $A^2 = I$, where I denote the identity matrix. Let $C^{m\times n}$ denote the set of all $m\times n$ matrices over complex field C, let $C^* = C\backslash\{0\}$. For $B \in C^{m\times n}$, we write σ(B) and |σ(B)| for the set of all distinct eigenvalues of B and the cardinality of σ(B), respectively. Let A and B be two complex matrices of order n. The linear combination of A and B is the matrix T = c_1A + c_2B, where $c_1, c_2 \in C^*$.

For any non-zero integers p and q, $p \mid q$ means that p is a divisor of q. For a complex number ε, let $\bar{\varepsilon}$ denote the conjugate of ε.

Lemma 1. [9] Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \{x \mid x^k = 1, x = \in C^*\}$, $\lambda_i \neq \lambda_j$ if $i \neq j$, and k is a positive integer. The following statements hold:

(i) $$\text{If } \lambda_1 = \lambda_2 + \lambda_3, \quad \text{then} \quad 6 \mid k \quad \text{and} \quad \{\lambda_2, \lambda_3\} = \{\varepsilon\lambda_1, \bar{\varepsilon}\lambda_1\}$$

(ii) $$\text{If } \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4, \quad \text{then} \quad \lambda_1 + \lambda_2 = 0$$

(iii) $$\text{If } \lambda_1 + \lambda_2 = 0, \quad \text{then} \quad 2 \mid k$$

(iv) $$2\lambda_1 = \lambda_2 + \lambda_3$$

where $\varepsilon = 1/2 - \sqrt{3}\mu/2,\ \mu^2 = -1$.

Lemma 2. [10] Let $A_2, B_2 \in C^{m\times n}$ be two idempotent matrices, $c_1, c_2 \in C^*$. If $A_2B_2 = B_2A_2$, then T = c_1A + c_2B is involutive if and only one of the following holds:

① (c_1, c_2) = (±1, ±1) and $A_2 + B_2 = I$,

② (c_1, c_2) = (1, –2) or (c_1, c_2) = (–1, 2) and $A_2 = I$,

③ (c_1, c_2) = (2, –1) or (c_1, c_2) = (–2, 1) and $B_2 = I$

Let I_n denote the identity matrix of order n. It is well-know that a (k + 1)-potent matrix B is diagonalizable, see [11], *i.e*, there exists an invertible matrix P such that:

$$B = P\left(\beta_1 I_{n_1} \oplus \cdots \oplus \beta_m I_{n_m}\right)P^{-1}$$

where $\{\beta_1, \cdots, \beta_m\} = \sigma(B) \subset \{0\} \cup V, V = \{x \mid x^k = 1, x \in C^*\}$.

Let:

$$E_i = P(0 \oplus \cdots \oplus I_{n_i} \oplus \cdots \oplus 0)P^{-1},\ I = 1, \ldots, m$$

Then $B = \beta_1 E_1 + \beta_2 E_2 + \ldots \beta_m E_m$. The following reasoning can also be found in [10].

Let A be a square matrix with the same order as B. If $AB \neq BA$, then there exists $i \in \{1,\ldots, m\}$ such that $AE_i \neq E_iA$. Without loss of generality, suppose that $AE_i \neq E_iA$ for $i = 1,\ldots, r$, $AE_i = E_iA$ for $i = r + 1,\ldots, m$. So we have:

$$\mathrm{B}=\sum_{i=1}^{r}\beta_i E_i+\sum_{i=r+1}^{m}\beta_i E_i=\mathrm{P}\left(\mathrm{B}_1\oplus\mathrm{B}_2\right)\mathrm{P}^{-1}$$

where

$$\mathrm{B}_1=\beta_1 I_{n_1}\oplus\cdots\oplus\beta_r I_{n_r},\quad \mathrm{B}_2=\beta_{r+1}I_{n_{r+1}}\oplus\cdots\oplus\beta_m I_{n_m} \tag{1}$$

Assume that

$$\mathrm{A}=\mathrm{P}\begin{pmatrix}\mathrm{A}_1 & X\\ Y & \mathrm{A}_2\end{pmatrix}\mathrm{P}^{-1}$$

where A_1 and B_1 are square matrices of the same order. Since $AE_i = E_iA (i = r + 1,\ldots, m)$, we have:

$$\mathrm{A}\left(\sum_{i=r+1}^{m}E_i\right)=\left(\sum_{i=r+1}^{m}E_i\right)\mathrm{A}$$

Hence we get $X = 0$, $Y = 0$, *i.e.*:

$$\mathrm{A}=\mathrm{P}\left(\mathrm{A}_1\oplus\mathrm{A}_2\right)\mathrm{P}^{-1} \tag{2}$$

By $\left(\mathrm{A}_1\oplus\mathrm{A}_2\right)\left(0\oplus\mathrm{B}_2\right)=\left(0\oplus\mathrm{B}_2\right)\left(\mathrm{A}_1\oplus\mathrm{A}_2\right)$, we get $\mathrm{A}_2\mathrm{B}_2=\mathrm{B}_2\mathrm{A}_2$.

In this case, the idempotency of matrix A and $c_1A + c_2B$ are equivalent to the idempotency of A_i and $c_1A_i + c_2B_i$ (i = 1, 2), the (k+1)-potency of B is equivalent to the (k+1)-potency of B_i (i = 1, 2). The proofs of the following three lemmas is similar to the proofs of some lemmas in [10].

Lemma 3. Let A, B $\in C^{m\times n}$ be two non-zero matrices and $A^2 = A$, $B^{k+1} = B$, $AB \neq BA$, let $T = c_1A + c_2B$, where $c_1, c_2 \in C^*$. Let B_1 be the matrix defined in eq. (1). For any $\beta\in\sigma(B_1)$, if T is involutive, then there exists an unique $\beta'\in\sigma(B_1)$ ($\beta' = \beta$) such that $c_1 + c_2(\beta + \beta') = 0$.

Proof. Without loss of generality, suppose that $\beta = \beta_1$, β_1 is defined in eq. (1). By eqs. (1) and (2), there exists an invertible matrix P such that:

$$\mathrm{A}=\mathrm{P}\begin{bmatrix}\mathrm{A}_1 & 0\\ 0 & \mathrm{A}_2\end{bmatrix}\mathrm{P}^{-1}=\mathrm{P}\begin{bmatrix}\mathrm{A}_{11} & \cdots & \mathrm{A}_{1r} & 0\\ \vdots & \ddots & \vdots & \vdots\\ \mathrm{A}_{r1} & \cdots & \mathrm{A}_{rr} & 0\\ 0 & \cdots & \cdots & \mathrm{A}_2\end{bmatrix}\mathrm{P}^{-1}$$

where A_1 and A_2 are the matrices defined in eq. (2), A_{ii} is a matrix of order $n_i (i = 1, 2,\ldots, r)$. Since $A_1B_1 \neq B_1A_1$, there exists $j \in \{2,\ldots, r\}$ such that $A_{1j} \neq 0$ or $A_{j1} \neq 0$. Suppose that $A_{1j} \neq 0$ (the case of $A_{1j} \neq 0$ can be proved in a similar way). If $T = c_1A + c_2B$ is involutive, then $c_1A_1 + c_2B_1$ is incolutive. Hence:

$$c_1^2\mathrm{A}_1 + c_2^2\mathrm{B}_1^2 + c_1c_2(\mathrm{A}_1\mathrm{B}_1 + \mathrm{B}_1\mathrm{A}_1) = \mathrm{I} \qquad (3)$$

Comparing the (1, *j*)-block of both sides of eq. (3), we get:

$$c_1^2\mathrm{A}_{1j} + c_1c_2(\beta_1 + \beta_2)\mathrm{A}_{1j} = 0$$

Due to the facts that $c_1 \neq 0$ and $\mathrm{A}_{1j} \neq 0$, we get $c_1 + c_2(\beta_1 + \beta_2) = = 0$. For $\beta_l \in \sigma(\mathrm{B}_1)$ $(l \geq 2)$, if $c_1 + c_2(\beta_1 + \beta_2) = 0$, then $c_2(\beta_j + \beta_l) = 0$, $\beta_j = \beta_l$.

Lemma 4. Let A, B ∈ $C^{m\times n}$ be two non-zero matrices and $\mathrm{A}^2 = \mathrm{A}$, $\mathrm{B}^{k+1} = \mathrm{B}$, and AB ≠ BA, let $\mathrm{T} = c_1\mathrm{A} + c_2\mathrm{B}$, where $c_1, c_2 \in C^*$. Let B_1 be the matrix defined in eq. (1). If T is involutive, then one of the following holds:

(a) $$\sigma(\mathrm{B}_1) = \{\lambda_1, \lambda_2\}, \quad \lambda_1 \neq \lambda_2, \quad c_1 + c_2(\lambda_1 + \lambda_2) = 0$$

(b) $$\sigma(B_1) = \{0, \lambda, \varepsilon\lambda, \bar{\varepsilon}\lambda\}, \quad c_1 + c_2\lambda = 0, \quad 6 \mid k \quad \text{and} \quad \lambda \in \{x \mid x^k = 1, x \in C^*\}$$

where $\varepsilon = 1/2 - \sqrt{3}\mu/2, \mu^2 = -1$.

Proof. Form *Lemma 3*, there exist two distinct eigenvalues $\lambda_1, \lambda_2 \in \sigma(\mathrm{B}_1)$ such that $c_1 + c_2(\lambda_1 + \lambda_2) = 0$, and $|\sigma(\mathrm{B}_1)| \geq 2$ is even. If $|\sigma(\mathrm{B}_1)| = 2$, part (a) holds.

If $|\sigma(\mathrm{B}_1)| > 3$, there exists $\lambda_3 \in \sigma(\mathrm{B}_1)\backslash\{\lambda_1, \lambda_2\}$. By *Lemma 3.* there exists $\lambda_4 \in \sigma(\mathrm{B}_1)\backslash\{\lambda_1, \lambda_2, \lambda_3\}$ such that $c_1 + c_2(\lambda_3 + \lambda_4) = 0$. Hence $\lambda_3 + \lambda_4 = \lambda_1 + \lambda_2$. If $0 \notin \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, by *Lemma 1*, we have $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = 0$, $c_1 = 0$, a contradiction with $c_1 \in C^*$. Hence $0 \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. By *Lemma 1* and *Lemma 3*, there exists $\lambda \in \sigma(\mathrm{B}_1)$ such that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = = \{0, \lambda, \varepsilon\lambda, \bar{\varepsilon}, \lambda\}, c_1 + c_2\lambda = 0$ and 6 | *k*. Suppose that there exists $\lambda_5 \in \sigma(\mathrm{B}_1)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, from *Lemma 3*, there exists $\lambda_6 \in \sigma(\mathrm{B}_1)$ with $\lambda_5 \neq \lambda_6$ such that $c_1 + c_2(\lambda_5 + \lambda_6) = 0$. By $c_1 + c_2\lambda = 0$ we get $\lambda = \lambda_5 + \lambda_6$. From *Lemma 1* we have $\lambda_5 \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, a contradiction with $\lambda_5 \in \sigma(\mathrm{B}_1)\backslash\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Hence part (b) holds when $|\sigma(\mathrm{B}_1)| > 3$.

Lemma 5. Let:

$$\mathrm{S} = \begin{bmatrix} \mathrm{X} & \mathrm{Y} \\ \mathrm{Z} & \mathrm{T} \end{bmatrix} \in C^{n\times n}, \quad \mathrm{W} = \begin{bmatrix} \beta_1\mathrm{I} & 0 \\ 0 & \beta_2\mathrm{I} \end{bmatrix} \in C^{n\times n}$$

be two non-zero matrices and $\mathrm{S}^2 = \mathrm{S}$, $\mathrm{W}^{k+1} = \mathrm{W}$, SW ≠ WS. Then $c_1\mathrm{S} + c_2\mathrm{W}$ $(c_1, c_2 \in C^*)$ is involution if and only if:

$$c_1 + c_2(\beta_1 + \beta_2) = 0, \quad \mathrm{X} = \frac{c_2^2\beta_1^2 - 1}{c_1c_2(\beta_2 - \beta_1)}\mathrm{I}, \quad \mathrm{T} = \frac{c_2^2\beta_2^2 - 1}{c_1c_2(\beta_1 - \beta_2)}\mathrm{I}$$

where $\beta_1 \neq \beta_2, \beta_1 \in V, \beta_2 \in V, V = \{x \mid x^k = 1, x \in C^*\}$.

Proof. By calculating, the given conditions are sufficient. We will show that they are necessary. If $c_1\mathrm{S} + c_2\mathrm{W}$ is an involutive matrix, we have:

$$c_1^2S + c_2^2W^2 + c_1c_2(SW + WS) - I = 0$$

$$c_1\left(c_1 + 2c_2\beta_1\right)X + (c_2^2\beta_1^2 - 1)I = 0 \qquad (4)$$

$$c_1\left[c_1 + c_2(\beta_1 + \beta_2)\right]Y = 0 \qquad (5)$$

$$c_1\left[c_1+c_2(\beta_1+\beta_2)\right]Z=0 \tag{6}$$

$$c_1\left(c_1+2c_2\beta_2\right)T+(c_2^2\beta_2^2-1)I=0 \tag{7}$$

From eqs. (5) and (6), we get $c_1 + c_2(\beta_1 + \beta_2) = 0$. Thus $c_1 = -c_2(\beta_1 + \beta_2)$. Substitute $c_1 = -c_2(\beta_1 + \beta_2)$ to eqs. (4) and (7), we have:

$$\mathrm{X}=\frac{c_2^2\beta_1^2-1}{c_1c_2(\beta_2-\beta_1)}\mathrm{I},\quad \mathrm{T}=\frac{c_2^2\beta_2^2-1}{c_1c_2(\beta_1-\beta_2)}\mathrm{I}$$

Main results

In this section, the sufficient and necessary conditions that $\mathrm{T} = c_1\mathrm{A} + c_2\mathrm{B}$ is an involutive matrix are given, where A is an idempotent matrix, B is a (*k*+1)-potent matrix, and $\mathrm{AB} \neq \mathrm{BA}$.

Theorem 1. Let $\mathrm{A}, \mathrm{B} \in C^{m\times n}$ be two non-zero matrices, $\mathrm{A}^2 = \mathrm{A}$, $\mathrm{B}^{k+1} = \mathrm{B}$, $\mathrm{AB} \neq \mathrm{BA}$, let A_1 and B_1 be the matrices defined in eqs. (2) and (1), respectively. Let $\mathrm{T}_1= c_1\mathrm{A}_1 + c_1\mathrm{B}_1$, where $c_1, c_2 \in C^*$. Then T_1 involutive if and only if one of following holds:

(i)
$$\mathrm{A}_1=\begin{bmatrix}\omega_1 I & Y\\ Z & \omega_2 I\end{bmatrix},\quad \mathrm{B}_1=\begin{bmatrix}\lambda_1 I & 0\\ 0 & \lambda_2 I\end{bmatrix}$$

where

$$\omega_1=\frac{c_2^2\lambda_1^2-1}{c_1c_2(\lambda_2-\lambda_1)},\quad \omega_2=\frac{c_2^2\lambda_2^2-1}{c_1c_2(\lambda_1-\lambda_2)},\quad \lambda_1\in V,\quad \lambda_2\in V,\quad V=\{x\,|\,x^k=1\},$$
$$\lambda_1\neq\lambda_2,\quad \text{and}\quad c_1+c_2(\lambda_1+\lambda_2)=0$$

(ii)
$$\mathrm{A}_1=\begin{bmatrix}\omega_1\mathrm{I} & \mathrm{Y}_1 & 0 & 0\\ \mathrm{Z}_1 & \omega_2\mathrm{I} & 0 & 0\\ 0 & 0 & \omega_3\mathrm{I} & \mathrm{Y}_2\\ 0 & 0 & \mathrm{Z}_2 & \omega_4\mathrm{I}\end{bmatrix},\quad \mathrm{B}_1=\begin{bmatrix}\varepsilon\lambda\mathrm{I} & 0 & 0 & 0\\ 0 & \bar{\varepsilon}\lambda\mathrm{I} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda\mathrm{I}\end{bmatrix},\quad c_1+c_2\lambda=0$$

where

$$\omega_1=\frac{c_2^2(\varepsilon\lambda)^2-1}{c_1c_2(\bar{\varepsilon}\lambda-\varepsilon\lambda)},\quad \omega_2=\frac{c_2^2(\bar{\varepsilon}\lambda)^2-1}{c_1c_2(\varepsilon\lambda-\bar{\varepsilon}\lambda)},\quad \omega_3=-\frac{1}{c_1c_2\lambda},\quad \omega_4=\frac{1-c_2^2\lambda^2}{c_1c_2\lambda},$$
$$\varepsilon=\frac{1}{2}-\frac{\sqrt{3}}{2}\mu,\quad \mu^2=-1,\quad \lambda\in\{x\,|\,x^k=1\}$$

Proof. By calculating, the given conditions are sufficient. We will show that they are necessary. If $c_1\mathrm{A}_1 + c_1\mathrm{B}_1$ is involutive, we have:

$$c_1^2\mathrm{A}_1+c_2^2\mathrm{B}_1^2+c_1c_2(\mathrm{A}_1\mathrm{B}_1+\mathrm{B}_1\mathrm{A}_1)-\mathrm{I}=0 \tag{8}$$

By *Lemma 4,* we only need to consider the following two cases:

Case 1. $\mathrm{B}_1 = \begin{bmatrix} \lambda_1 \mathrm{I} & 0 \\ 0 & \lambda_2 \mathrm{I} \end{bmatrix}$ and $\lambda_1 \neq \lambda_2, c_1 + c_2(\lambda_1 + \lambda_2) = 0.$

Suppose that $\mathrm{A}_1 = \begin{bmatrix} \mathrm{X} & \mathrm{Y} \\ \mathrm{Z} & \mathrm{T} \end{bmatrix}$. From *Lemma 5*, we get:

$$\mathrm{A}_1 = \begin{bmatrix} \omega_1 \mathrm{I} & \mathrm{Y} \\ \mathrm{Z} & \omega_2 \mathrm{I} \end{bmatrix}, \quad \omega_1 = \frac{c_2^2 \lambda_1^2 - 1}{c_1 c_2 (\lambda_2 - \lambda_1)}, \quad \omega_2 = \frac{c_2^2 \lambda_2^2 - 1}{c_1 c_2 (\lambda_1 - \lambda_2)}$$

Case 2. $\mathrm{B}_1 = \begin{bmatrix} \varepsilon\lambda \mathrm{I} & 0 & 0 & 0 \\ 0 & \bar{\varepsilon} I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda \mathrm{I} \end{bmatrix}, \quad c_1 + c_2 \lambda = 0 \quad \text{and} \quad 6 \mid k.$

Suppose that

$$\mathrm{A}_1 = \begin{bmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} & \mathrm{X}_{13} & \mathrm{X}_{14} \\ \mathrm{X}_{21} & \mathrm{X}_{22} & \mathrm{X}_{23} & \mathrm{X}_{24} \\ \mathrm{X}_{31} & \mathrm{X}_{32} & \mathrm{X}_{33} & \mathrm{X}_{34} \\ \mathrm{X}_{41} & \mathrm{X}_{42} & \mathrm{X}_{43} & \mathrm{X}_{44} \end{bmatrix}, \quad \text{let} \quad \beta_1 = \varepsilon\lambda, \quad \beta_2 = \bar{\varepsilon}\lambda, \quad \beta_3 = 0, \quad \beta_4 = \lambda$$

Comparing (i, j)-block $(i \neq j)$ in eq. (8) ,we get:

$$c_1^2 \mathrm{X}_{ij} + c_1 c_2 (\beta_i + \beta_j) \mathrm{X}_{ij} = 0, \quad [c_1 + c_2(\beta_i + \beta_j)] \mathrm{X}_{ij} = 0$$

By $c_1 + c_2\lambda = 0$, we can get $c_2(\beta_i + \beta_j - \lambda)\mathrm{X}_{ij} = 0$. If $\beta_i + \beta_j \neq \lambda$, then $\mathrm{X}_{ij} = 0$. Thus:

$$\mathrm{A}_1 = \begin{bmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} & 0 & 0 \\ \mathrm{X}_{21} & \mathrm{X}_{22} & 0 & 0 \\ 0 & 0 & \mathrm{X}_{33} & \mathrm{X}_{34} \\ 0 & 0 & \mathrm{X}_{43} & \mathrm{X}_{44} \end{bmatrix} = \begin{bmatrix} \mathrm{P}_1 & 0 \\ 0 & \mathrm{P}_2 \end{bmatrix}, \quad \mathrm{P}_1 = \begin{bmatrix} \mathrm{X}_{11} & \mathrm{X}_{12} \\ \mathrm{X}_{21} & \mathrm{X}_{22} \end{bmatrix}, \quad \mathrm{P}_2 = \begin{bmatrix} \mathrm{X}_{33} & \mathrm{X}_{34} \\ \mathrm{X}_{43} & \mathrm{X}_{44} \end{bmatrix}$$

Let:

$$\mathrm{Q}_1 = \begin{bmatrix} \varepsilon\lambda \mathrm{I} & 0 \\ 0 & \bar{\varepsilon}\lambda \mathrm{I} \end{bmatrix}, \quad \mathrm{Q}_2 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \mathrm{I} \end{bmatrix}$$

then $\mathrm{B}_1 = \mathrm{Q}_1 \oplus \mathrm{Q}_2$. Since $c_1\mathrm{A}_1 + c_2\mathrm{B}_1$ is involutive, we have $c_1\mathrm{P}_1 + c_2\mathrm{Q}_1$ and $c_1\mathrm{P}_2 + c_2\mathrm{Q}_2 = 0$ are involutive. Form *Lemma 5*, we get:

$$\mathrm{P}_1 = \begin{bmatrix} \omega_1 \mathrm{I} & \mathrm{Y}_1 \\ \mathrm{Z}_1 & \omega_2 \mathrm{I} \end{bmatrix}, \quad \mathrm{P}_2 = \begin{bmatrix} \omega_3 \mathrm{I} & \mathrm{Y}_2 \\ \mathrm{Z}_2 & \omega_4 \mathrm{I} \end{bmatrix}, \quad \omega_1 = \frac{c_2^2 (\varepsilon\lambda)^2 - 1}{c_1 c_2 (\bar{\varepsilon}\lambda - \varepsilon\lambda)}, \quad \omega_2 = \frac{c_2^2 (\bar{\varepsilon}\lambda)^2 - 1}{c_1 c_2 (\varepsilon\lambda - \bar{\varepsilon}\lambda)},$$

$$\omega_3 = -\frac{1}{c_1 c_2 \lambda}, \quad \omega_4 = \frac{1 - c_2^2 \lambda^2}{c_1 c_2 \lambda}$$

For an eigenvalue u of a square matrix, let $m(u)$ denote the algebraic multiplicity of u.

Theorem 2. Let A, B $\in C^{m\times n}$ be two non-zero matrices, $A^2 = A$, $B^{k+1} = B$, $AB \neq BA$, let A_2 and B_2 be the matrices defined in eqs. (2) and (1), respectively. Let $T_2 = c_1A_2 + c_2B_2$, where $c_1, c_2 \in C^*$. Then T_2 is involutive if and only if one of the following holds:

(a) $$c_2 = \pm\frac{1}{u},\quad A_2 = 0,\quad \{u\}\subseteq\sigma(B_2)\subseteq\{u,-u\},\quad \text{where}\quad u\in\{x\,|\,x^k = 1\}$$

(b) $$c_1 + c_2u = \pm 1,\quad A_2 = I,\quad B_2 = uI,\quad \text{where}\quad u\in\{x\,|\,x^k = 1\}\bigcup\{0\}$$

(c) $$(c_1, c_2) = \left(\pm 2, \mp\frac{1}{u}\right) \text{and } \sigma(A_2) = \{1, 0\},\quad B_2 = uI,\quad \text{where}\quad u\in\{x\,|\,x^k = 1\}$$

(d) $(c_1, c_2) = \left(\mp\frac{u+v}{u-v}, \pm\frac{2}{u-v}\right)$, and $A_2 = I$, $\sigma(B_2) = \{u, v\}$, where $u, v\in\{x\,|\,x^k = 1\}\bigcup\{0\}$, $u \neq v$

(e) $$(c_1, c_2) = \left(\mp\frac{u-v}{v}, \pm\frac{1}{v}\right),\quad \text{or}\quad (c_1, c_2) = \left(\mp\frac{v+u}{v}, \mp\frac{1}{v}\right)$$

and there exists an invertible matrix P_2 such that:

$$A_2 = P_2(I\oplus 0)P_2^{-1},\quad B_2 = P_2(uI\oplus vI)P_2^{-1}$$

where $u\in\{x\,|\,x^k = 1\}\bigcup\{0\}$, $v\in\{x\,|\,x^k = 1\}$, $m(u) = rank(A)$.

(f) $$(c_1, c_2) = \left(\mp\frac{u-v}{v}, \pm\frac{1}{v}\right),\quad \text{or}\quad (c_1, c_2) = \left(\mp\frac{v+u}{v}, \mp\frac{1}{v}\right)$$

and there exists an invertible matrix P_2 such that:

$$A_2 = P_2(I\oplus 0)P_2^{-1},\quad B_2 = P_2[uI\oplus(vI\oplus -vI)]P_2^{-1},$$

where $u\in\{x\,|\,x^k = 1\}\bigcup\{0\}$, $v\in\{x\,|\,x^k = 1\}$, $m(u) = rank(A)$.

Proof. By calculating, the given conditions are sufficient. We will show that they are necessary. Firstly we consider the case that A_2 and B_2 are two linearly dependent matrices.

If $A_2 = 0$, then $c_2 \in \{-1/u, 1/u\}$, part (a) holds.

If $B_2 = 0$, we have $c_1 = \pm 1$. This is a special case of part (b)

If $B_2 = uA_2 (u \in C^*)$, by A_2 is an idempotent matrix and T_2 is an involutive matrix, we get $c_1 + c_2u \in \{-1, 1\}$, part (b) holds.

Next we only consider the case that A_2 and B_2 are two linearly independent matrices.

If B_2 has only one non-zero eigenvalue u, then B_2/u is idempotent. Therefore:

$$T_2 = c_1A_2 + c_2B_2 = c_1A_2 + c_2u\frac{B_2}{u}$$

By *Lemma 2* we know that one of the following holds:

① $$c_1 = \pm 1,\quad c_2u = \pm 1,\quad A_2 + \frac{1}{u}B_2 = I$$

② $$c_1=1,\quad c_2u=-2,\quad \text{or}\quad c_1=-1,\quad c_2u=2,\quad \mathrm{A}_2=\mathrm{I}$$

③ $$c_1=2,\quad c_2u=-1,\quad \text{or}\quad c_1=-2,\quad c_2u=1,\quad \frac{1}{u}\mathrm{B}_2=\mathrm{I}$$

From ①-③ we know that one of parts (c) and (d) holds.

Next we only consider the case that B_2 has at least two non-zero eigenvalues. Since $\mathrm{A}_2^2=\mathrm{A}_2$, $\mathrm{B}_2^{k+1}=\mathrm{B}_2$, and $\mathrm{A}_2\mathrm{B}_2=\mathrm{B}_2\mathrm{A}_2$, there exists a non-singular matrix P_2 such that:

$$\mathrm{A}_2=\mathrm{P}_2\begin{pmatrix}\mathrm{I}_r & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\mathrm{P}_2^{-1},\quad \mathrm{B}_2=\mathrm{P}_2\begin{pmatrix}\beta_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \beta_p\end{pmatrix}\mathrm{P}_2^{-1}$$

where $r=rank(\mathrm{A}_2)$, $\beta_i\in\{0\}\cup\mathrm{V}$, $\mathrm{V}=\{x\,|\,x^k=1\}$, $i=1,2,\cdots,p.$

Suppose that:

$$\mathrm{T}_2=\mathrm{P}_2\begin{pmatrix}\gamma_1 & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \gamma_p\end{pmatrix}\mathrm{P}_2^{-1}$$

Since $\mathrm{T}_2 = c_1\mathrm{A}_2 + c_2\mathrm{B}_2$ and $\mathrm{T}_2^2=\mathrm{I}$, we have $\gamma_1,\ldots,\gamma_p\in\{-1, 1\}$ and

$$\begin{aligned} &c_1+c_2\beta_m=\gamma_m \quad && m=1,2,\cdots,r\\ &c_2\beta_s=\gamma_s && s=r+1,\cdots,p\end{aligned} \tag{9}$$

Note that $\mathrm{A}_2\neq 0$ and B_2 has at lest two non-zero eigenvalues. We can consider the following cases:

Case 1. If $\beta_1\neq 0,\beta_2\neq 0,\beta_1\neq\beta_2$, and $\gamma_1\neq\gamma_2$, then:

$$c_1=\frac{-\beta_1-\beta_2}{\beta_1-\beta_2},\quad c_2=\frac{2}{\beta_1-\beta_2},\quad \text{or}\quad c_1=\frac{\beta_1+\beta_2}{\beta_1-\beta_2},\quad c_2=\frac{-2}{\beta_1-\beta_2}$$

Form eq. (9), we get:

$$\left(\frac{-\beta_1-\beta_2}{\beta_1-\beta_2}+\frac{2}{\beta_1-\beta_2}\beta_m\right)\in\{-1,1\},\quad m=3,\cdots,r,\quad \text{and}\quad \frac{2}{\beta_1-\beta_2}\beta_s\in\{-1,1\},\quad s=r+1,\cdots,p$$

By:

$$\left(\frac{-\beta_1-\beta_2}{\beta_1-\beta_2}+\frac{2}{\beta_1-\beta_2}\beta_m\right)\in\{-1,1\},\quad \text{we have}\quad \beta_m=\beta_1\quad \text{or}\quad \beta_m=\beta_2,\quad m=3,\cdots,r$$

Since:

$$c_1\neq 0,\quad \frac{2}{\beta_1-\beta_2}\beta_s\in\{-1,1\},\quad \text{and}\quad \beta_1,\beta_2\in\left\{x\,|\,x^k=1\right\}$$

we have $r = p$, and when $r = p$, $\mathrm{A}_2 = \mathrm{I}_n$, $\sigma(\mathrm{B}_2) = \{\beta_1,\beta_2\}$. Hence part (d) holds.

Case 2. If $\beta_1\neq 0,\beta_2\neq 0,\beta_1\neq\beta_2$, and $\gamma_1=\gamma_2$, then $c_2 = 0$, a contradiction with $c_2\in C^*$.

Case 3. If $\beta_1\neq 0,\beta_{r+1}\neq 0,\beta_1\neq\beta_{r+1}$, and $\gamma_1=\gamma_{r+1}$, then:

$$c_1=\frac{\beta_1+\beta_{r+1}}{\beta_{r+1}},\quad c_2=-\frac{1}{\beta_{r+1}},\quad c_1=\frac{-\beta_1-\beta_{r+1}}{\beta_{r+1}},\quad c_2=\frac{1}{\beta_{r+1}}$$

From eq. (9), we get:

$$\left(\frac{\beta_1+\beta_{r+1}}{\beta_{r+1}}-\frac{1}{\beta_{r+1}}\beta_m\right)\in\{-1,1\},\quad m=2,\cdots,r,\quad \text{and}\quad -\frac{1}{\beta_{r+1}}\beta_s\in\{-1,1\},\quad s=r+2,\cdots,p$$

Hence we have $\beta_m = \beta_1$, $m = 2,\ldots, r$, and $\beta_s = \pm\,\beta_{r+1}$, $s = r + 2,\ldots, p$. If $\beta_s = \beta_{r+1}$, then part (e) holds. If $\beta_s = -\beta_{r+1}$, then part (f) holds.

Case 4. If $\beta_1 \neq 0$, $\beta_{r+1} \neq 0$, $\beta_1 \neq \beta_{r+1}$, and $\gamma_1 = \gamma_{r+1}$, then:

$$c_1=\frac{\beta_{r+1}-\beta_1}{\beta_{r+1}},\quad c_2=\frac{1}{\beta_{r+1}},\quad \text{or}\quad c_1=\frac{-\beta_{r+1}+\beta_1}{\beta_{r+1}},\quad c_2=-\frac{1}{\beta_{r+1}}$$

From eq. (9), we get:

$$\left(\frac{\beta_{r+1}-\beta_1}{\beta_{r+1}}+\frac{1}{\beta_{r+1}}\beta_m\right)\in\{-1,1\},\quad m=2,\cdots,r,\quad \text{and}\quad \frac{1}{\beta_{r+1}}\beta_s\in\{-1,1\},\quad s=r+2,\cdots,p$$

Hence we have $\beta_m = \beta_1$, $m = 2,\ldots, r$, and $\beta_s = \pm\,\beta_{r+1}$, $s = r + 2,\ldots, p$. If $\beta_s = \beta_{r+1}$, then part (e) holds. If $\beta_s = -\beta_{r+1}$, then part (f) holds.

Case 5. If $\beta_{r+1} \neq 0$, $\beta_{r+2} \neq 0$, $\beta_{r+1} \neq \beta_{r+2}$, and $\gamma_{r+1} = \gamma_{r+2}$, then:

$$c_2=\frac{1}{\beta_{r+1}}=-\frac{1}{\beta_{r+2}},\quad \text{or}\quad c_2=-\frac{1}{\beta_{r+1}}=\frac{1}{\beta_{r+2}},\quad \text{hence}\quad \beta_{r+1}=-\beta_{r+1}$$

From eq. (9), we get:

$$\left(c_1+\frac{1}{\beta_{r+1}}\beta_m\right)\in\{-1,1\},\quad m=1,\cdots,r,\quad \text{and}\quad \frac{1}{\beta_{r+1}}\beta_s\in\{-1,1\},\quad s=r+3,\cdots,p$$

Hence $\beta_m = (-1 - c_1)\beta_{r+1}$ or, $\beta_m = (1 - c_1)\beta_{r+1}$, $m = 1,\ldots, r$, and $\beta_s = \pm\,\beta_{r+1}$, $s = r + 3,\ldots, p$. So we have $c_1 = \pm 1$, $\beta_m = 0$. In this case, part (f) holds.

Case 6. If β_r+1 $\neq 0$, $\beta_{r+2} \neq 0$, $\beta_{r+1} \neq \beta_{r+2}$, and $\gamma_{r+1} = \gamma_{r+1}$, then:

$$c_2=\frac{1}{\beta_{r+1}}=\frac{1}{\beta_{r+2}},\quad c_2=\frac{-1}{\beta_{r+1}}=\frac{-1}{\beta_{r+2}},$$

a contradiction to $\beta_{r+1} \neq \beta_{r+2}$.

Therorem 3.3. Let A, B $\in C^{n\times n}$ be two non-zero matrices, $A^2 = A$, $B^{k+1} = B$, AB ≠ BA, let B_2 and B_2 be the matrices defined in eq. (1). Let $T = c_1A_2 + c_2B_2$, where $c_1, c_2 \in C^*$. Then T is involutive if and only if one of the following holds:

① Part (a) of *Theorem 2* and part (i) of *Theorem 1* hold:

$$(c_1,c_2)=\left(\mp\frac{\lambda_1+\lambda_2}{u},\pm\frac{1}{u}\right)$$

② Part (a) of *Theorem 2* and part (ii) of *Theorem 1* hold:

$$(c_1,c_2)=\left(\mp\frac{\lambda}{u},\pm\frac{1}{u}\right)$$

③ Part (b) of *Theorem 2* and part (i) of *Theorem 1* hold:

$$(c_1, c_2) = \left(\pm \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - u}, \mp \frac{1}{\lambda_1 + \lambda_2 - u} \right)$$

④ Part (b) of *Theorem 2* and part (ii) of *Theorem 1* hold:

$$(c_1, c_2) = \left(\pm \frac{\lambda}{\lambda - u}, \mp \frac{1}{\lambda - u} \right)$$

⑤ Part (d) of *Theorem 2* and part (i) of *Theorem 1* hold:

$$(c_1, c_2) = \left(\mp \frac{2(\lambda_1 + \lambda_2)}{u - v}, \pm \frac{2}{u - v} \right)$$

⑥ Part (e) of *Theorem 2* and part (i) of *Theorem 1* hold:

$$(c_1, c_2) = \left(\mp \frac{\lambda_1 + \lambda_2}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\pm 1, \mp \frac{1}{u} \right), \quad \lambda_1 = \varepsilon u, \quad \lambda_2 = \varepsilon^{-1} u, \quad 6 \mid k$$

⑦ Part (e) of *Theorem 2* and part (ii) of *Theorem 1* hold:

$$(c_1, c_2) = \left(\mp \frac{\lambda}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\pm \varepsilon, \mp \frac{\varepsilon}{\lambda} \right), \quad u = \varepsilon \lambda, \quad v = \varepsilon^{-1} \lambda, \quad 6 \mid k$$

⑧ Part (f) of *Theorem 2* and part (i) of *Theorem 1* hold:

$$(c_1, c_2) = \left(\mp \frac{\lambda_1 + \lambda_2}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\mp 1, \pm \frac{1}{u} \right), \quad \lambda_1 = \varepsilon u, \quad \lambda_2 = \varepsilon^{-1} u, \quad 6 \mid k$$

⑨ Part (f) of *Theorem 2* and part (ii) of *Theorem 1* hold:

$$(c_1, c_2) = \left(\mp \frac{\lambda}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\pm \varepsilon, \mp \frac{\varepsilon}{\lambda} \right), \quad u = \varepsilon \lambda, \quad v = \varepsilon^{-1} \lambda, \quad 6 \mid k$$

where

$$\lambda_1, \lambda_2, \lambda \in \sigma(B_1), \quad u, v \in \sigma(B_2), \quad \varepsilon = \frac{1}{2} - \frac{\sqrt{3}}{2} \mu, \quad \mu^2 = -1$$

Proof. We only need to find the linear combination T = c_1A$_2$ + c_2B$_2$ that satisfy the conditions of *Theorems 1* and *Theorems 2*. Obviously the given conditions are sufficient. We will show that they are necessary. We can consider the following cases.

Case 1. Part (a) of *Theorem 2* holds. If part (i) of *Theorem 1* holds, by $c_1 + c_2(\lambda_1 + \lambda_2) = 0$, we have:

$$c_1 = \mp \frac{\lambda_1 + \lambda_2}{u}$$

If part (ii) of *Theorem 1* holds, by $c_1 + c_2\lambda = 0$, we have:

$$c_1 = \mp \frac{\lambda}{u}$$

In this case, part ① or ② holds.

Case 2. Part (b) of *Theorem 2* holds. If part (i) of *Theorem 1* holds, then:

$$(c_1, c_2) = \left(\pm \frac{\lambda_1 + \lambda_2}{\lambda_1 + \lambda_2 - u}, \mp \frac{1}{\lambda_1 + \lambda_2 - u} \right)$$

where $\lambda_1, \lambda_2 \in \sigma(B_1)$, $u \in \sigma(B_2)$.

If part (ii) of *Theorem 1* holds, then:

$$(c_1, c_2) = \left(\pm \frac{\lambda}{\lambda - u}, \mp \frac{1}{\lambda - u} \right)$$

where $\lambda \in \sigma(B_1)$, $u \in \sigma(B_2)$. In this case, part ③ or ④ holds.

Case 3. Part (c) of *Theorem 2* holds. If part (i) of *Theorem 1* holds, then $\lambda_1 + \lambda_2 = 2u$, a contradiction to *Lemma 1*. If part (ii) of *Theorem 1* holds, then $\lambda = 2u$, a contradiction to $\lambda, u = \in \{x \mid x^k = 1\}$.

Case 4. Part (d) of *Theorem 2* holds. If part (i) of *Theorem 1* holds, then:

$$(c_1, c_2) = \left(\mp \frac{2(\lambda_1 + \lambda_2)}{u - v}, \pm \frac{2}{u - v} \right)$$

If part (ii) of *Theorem 1* holds, then $u + v = 2\lambda$, a contradiction to *Lemma 1*. In this case, part ⑤ holds.

Case 5. Part (e) of *Theorem 2* holds. If part (i) of *Theorem 1* holds, by *Lemma 1*, we have:

$$(c_1, c_2) = \left(\mp \frac{\lambda_1 + \lambda_2}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\pm 1, \mp \frac{1}{u} \right), \quad \lambda_1 = \varepsilon u, \quad \lambda_2 = \varepsilon^{-1} u, \quad 6 \mid k$$

If part (ii) of *Theorem 1* holds, by *Lemma 1*, we have:

$$(c_1, c_2) = \left(\mp \frac{\lambda}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\pm \varepsilon, \mp \frac{\varepsilon}{\lambda} \right), \quad u = \varepsilon \lambda, \quad v = \varepsilon^{-1} \lambda, \quad , \quad 6 \mid k$$

In this case, part ⑥ or ⑦ holds.

Case 6. Part (f) of *Theorem 2* holds. If part (i) of *Theorem 1* holds, by *Lemma 1*, we have:

$$(c_1, c_2) = \left(\mp \frac{\lambda_1 + \lambda_2}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\mp 1, \pm \frac{1}{u} \right), \quad \lambda_1 = \varepsilon u, \quad \lambda_2 = \varepsilon^{-1} u, \quad 6 \mid k$$

If part (ii) of *Theorem 1* holds, by *Lemma 1*, we have:

$$(c_1, c_2) = \left(\mp \frac{\lambda}{v}, \pm \frac{1}{v} \right), \quad \text{or} \quad (c_1, c_2) = \left(\pm \varepsilon, \mp \frac{\varepsilon}{\lambda} \right), \quad u = \varepsilon \lambda, \quad v = \varepsilon^{-1} \lambda, \quad 6 \mid k$$

In this case, part ⑧ or ⑨ holds.

Conclusions

In conclusion, our exploration of idempotent matrices and their applications in MEMS has revealed a plethora of possibilities. The efficacy of idempotent matrices in the multiple-noise removal process for MEMS sensors represents a substantial breakthrough, holding considerable promise for enhancing the reliability and accuracy of these vital devices.

As research continues to delve deeper into the realm of matrix analysis for MEMS, it is reasonable to expect the emergence of even more innovative solutions. This research paves the way for future advancements in MEMS technology, inspiring engineers and researchers to push the boundaries and create more sophisticated and efficient MEMS systems.

The potential applications of idempotent matrices in MEMS and other fields like damage identification [12] and 3-D printing systems [13] are extensive and promising, ranging from enhancing sensor performance to facilitating the development of new micro-scale devices. Further research and development in this area is expected to ensure that MEMS technology [14] plays an even more integral role in our lives, powered by the power of idempotent matrices and advanced matrix analysis techniques.

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