ANALYTICAL SOLUTIONS FOR A CLASS OF FRACTAL KORTEWEG-de VRIES TYPE EQUATION

by

Wenze DONG [*]

School of General Education, Huanghe Jiaotong University, Wuzhi, China

Original scientific paper
https://doi.org/10.2298/TSCI2602879D

The present study focuses on a class of Korteweg-de Vries (KdV)-type equations involving the time-space fractal scaling law derivative. The objective of this investigation is to explore their exact analytical solutions. The employment of fractal scaling law derivatives, calculus theory, and Jacobi elliptic functions, in conjunction with variable substitutions and equation transformations, facilitates the attainment of precise analytical solutions for this equation type under various conditions. The findings of the research endeavor have yielded two notable outcomes. Firstly, they have augmented the solution system for KdV-type equations. Secondly, they have furnished an effective method reference for solving other non-linear fractal partial differential equations. These contributions are instrumental in fostering the advancement of the application of fractal calculus in the domain of mathematical physics.

Keywords: *fractal KdV-type equation, fractal scaling, law derivative, Jacobi elliptic function*

Introduction

The subsequent classical KdV equation was introduced by Korteweg and de Vries to describe shallow water waves of long wavelength with small amplitude [1-3]:

$$\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0 \tag{1}$$

It is among the most extensively studied equations in the field of mathematical physics in the present era. Consequently, a substantial corpus of literature has emerged that discusses a considerable number of non-linear KdV-type equations. A substantial body of research has been conducted by scientists and researchers concerning the precise traveling wave solutions of these equations. These investigations have been undertaken through the utilization of diverse methodologies, for examples, the variational method [4-7], the exp-function method [8, 9], tanh and sech function methods [10]. In recent years, fractal calculus has played a very important role in various application areas, such as modeling anomalous diffusion, heat transfer, seismic wave analysis, signal processing, control theory, and many other dynamical systems. The properties of several physical phenomena are found to be best described by fractal differential equations. In a recent study, the fractal scaling-law derivative was proposed as a means to investigate the relationships between the general calculus [11]

---

[*] Author's e-mail: dongwenze54@163.com

and fractal geometry [12]. A study was conducted on a series of models of applied science that were based on the fractal scaling-law derivative [13-20].

In the present work, we consider the following fractal KdV-type equation:

$$ {}^{MSL}\mathrm{D}_t^{(1)}u\left(x,t\right)+\left(qu+p\right)u\,{}^{MSL}\mathrm{D}_x^{(1)}u\left(x,t\right)+r\,{}^{MSL}\mathrm{D}_x^{(3)}u\left(x,t\right)=0 \tag{2} $$

where p, q, and r are constants and the symbols, and ${}^{MSL}D_t^{(1)}$ and ${}^{MSL}D_x^{(1)}$ – the time fractal scaling-law derivative and the space fractal scaling-law derivative, respectively [13, 14]. The objective of this study is to investigate a novel class of exact solutions to eq. (2) by employing the Jacobi elliptic functions.

Preliminaries

The fractal scaling-law derivative

In this section, we will review some fundamental definitions and properties of the fractal scaling-law derivative. For a more comprehensive overview of this subject, please refer to the works of [13, 14].

Definition 1. The time fractal scaling-law derivative of $u(x, t)$ is defined:

$$ {}^{MSL}\mathrm{D}_t^{(1)}u\left(x,t\right)=\alpha\left(\lambda,\mu,t\right)\frac{\partial}{\partial t}u\left(x,t\right) \tag{3} $$

where $\alpha(\lambda,\mu,t)=t^{\mu}\lambda^{-1}(1-\mu)^{-1}$, the constant $\lambda>0$, and the fractal dimension $0\le\mu\le1$.

Definition 2. The space fractal scaling-law derivative of $u(x, t)$ is defined:

$$ {}^{MSL}\mathrm{D}_x^{(1)}u\left(x,t\right)=\beta\left(\bar{\lambda},\bar{\mu},x\right)\frac{\partial}{\partial x}u\left(x,t\right) \tag{4} $$

where $\beta\left(\bar{\lambda},\bar{\mu},x\right)=x^{\bar{\mu}}\bar{\lambda}^{-1}\left(1-\bar{\mu}\right)^{-1}$, the constant $\bar{\lambda}>0$, and the fractal dimension $0\le\bar{\mu}\le1$.

Next, let us denote the scaling-law function space Ω by:

$$ \Omega=\left\{\Xi\left(\tau\right)\mid {}^{MSL}D_\tau^{(1)}\Xi\left(\tau\right)\ exists\right\} $$

Theorem 1 [13, 14]. Let v is an constant, $M\in\Omega$, and $N\in\Omega$. Then:

$$ {}^{MSL}\mathrm{D}_\tau^{(1)}\left(v\right)=0 \tag{5} $$

$$ {}^{MSL}\mathrm{D}_\tau^{(1)}\left(\lambda\tau^{1-\mu}\right)=1 \tag{6} $$

$$ {}^{MSL}\mathrm{D}_\tau^{(1)}\left[vM\left(\tau\right)\right]=v\,{}^{MSL}\mathrm{D}_\tau^{(1)}\left[M\left(\tau\right)\right] \tag{7} $$

$$ {}^{MSL}\mathrm{D}_\tau^{(1)}\left[M\left(\tau\right)+N\left(\tau\right)\right]={}^{MSL}\mathrm{D}_\tau^{(1)}\left[M\left(\tau\right)\right]+{}^{MSL}\mathrm{D}_\tau^{(1)}\left[N\left(\tau\right)\right] \tag{8} $$

$$ {}^{MSL}\mathrm{D}_\tau^{(1)}\left[M\left(\tau\right)N\left(\tau\right)\right]=N\left(\tau\right){}^{MSL}\mathrm{D}_\tau^{(1)}\left[M\left(\tau\right)\right]+M\left(\tau\right){}^{MSL}\mathrm{D}_\tau^{(1)}\left[N\left(\tau\right)\right] \tag{9} $$

Jacobi elliptic functions

In this section, we will briefly outline the primary characteristics of the Jacobi elliptic functions that will be utilized in this study. For a more comprehensive overview of these functions, please refer to [21, 22].

Let k be a number in (0, 1), and let ξ denote a real variable. The Jacobi elliptic functions $sn(\xi, k)$, $cn(\xi, k)$, and $dn(\xi, k)$ are defined as the solutions of the system of differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}\xi} = yz, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = -zx, \quad \frac{\mathrm{d}z}{\mathrm{d}\xi} = -k^2 xy \tag{10}$$

that satisfy the initial conditions:

$$sn(0,k) = x(0) = 0, \quad cn(0,k) = y(0) = 1, \quad dn(0,k) = z(0) = 1$$

The parameter k is known as the modulus and satisfies $0 < k < 1$.

Equations (10) are real analytic in the variables ξ, x, y, z and k, so the basic existence theorem of ODE ensures that the Jacobi elliptic functions are smooth.

The definition immediately gives the derivatives for the functions, namely:

$$\frac{\mathrm{d}}{\mathrm{d}\xi} sn(\xi,k) = cn(\xi,k) dn(\xi,k) \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} cn(\xi,k) = -sn(\xi,k) dn(\xi,k) \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} dn(\xi,k) = -k^2 cn(\xi,k) sn(\xi,k) \tag{13}$$

Theorem 2 [21]. As k approaches 0 from the right, we have:

$$sn(\xi,k) \to \sin(\xi), \quad cn(\xi,k) \to \cos(\xi), \quad dn(\xi,k) \to 1 \tag{14}$$

and as k approaches 1 from the left, we have:

$$sn(\xi,k) \to \tanh(\xi), \; cn(\xi,k) \to \operatorname{sech}(\xi), \; dn(\xi,k) \to \operatorname{sech}(\xi) \tag{15}$$

Theorem 3 [21]. For fixed $k, 0 < k < 1,$ the following identities hold:

$$sn^2(\xi,k) + cn^2(\xi,k) \equiv 1 \tag{16}$$

$$k^2 sn^2(\xi,k) + dn^2(\xi,k) \equiv 1 \tag{17}$$

The solutions of eq. (2)

In this section, by using the Jacobi elliptic functions, we seek the exact solutions of eq. (2).

Firstly, by the two-scale transform [23, 24]:

$$Y = \frac{\bar{\lambda}}{x^{\bar{\mu}-1}}, \quad Z = \frac{\lambda}{t^{\mu-1}} \tag{18}$$

Equation (2) is converted into the following form:

$$\frac{\partial u}{\partial Z}+\left(p+qu\right)u\frac{\partial u}{\partial Y}+r\frac{\partial^3 u}{\partial Y^3}=0 \tag{19}$$

Suppose eq. (19) has the traveling solution of the form:

$$u\left(Y,Z\right)=u\left(\xi\right),\quad \xi=\varepsilon Y+\omega Z \tag{20}$$

where ε and ω are constants.

Substituting eq. (20) into eq. (19) yields:

$$\omega u_\xi+p\varepsilon u u_\xi+q\varepsilon u^2 u_\xi+r\varepsilon^3 u_{\xi\xi\xi}=0 \tag{21}$$

Integrating both sides of eq. (21) with respect to ξ, we obtain:

$$6\omega u+3p\varepsilon u^2+2q\varepsilon u^3+6r\varepsilon^3 u_{\xi\xi}=c_1$$

Let $u_\xi=v.$ Then we get:

$$6\omega u^2+2p\varepsilon u^3+q\varepsilon u^4+6r\varepsilon^3 v^2=c_1 u+c_2 \tag{22}$$

where c_1 and c_2 are the integration constants.

Now, if $q=0$, we seek solutions of eq. (22) in the form:

$$u\left(\xi\right)=\frac{A_1+A_2 sn^2\left(\xi\right)}{b_1+b_2 sn\left(\xi\right)} \tag{23}$$

where A_1, A_2, b_1 and b_2 are constants to be determined.

Using eqs. (11)-(13), we get:

$$u_\xi=\frac{\left[2A_2 b_1 sn\left(\xi\right)-A_1 b_2\right]cn\left(\xi\right)dn\left(\xi\right)+A_2 b_2 sn^2\left(\xi\right)cn\left(\xi\right)dn\left(\xi\right)}{\left[b_1+b_2 sn\left(\xi\right)\right]^2} \tag{24}$$

We firstly take $A_2=0.$

By eq. (16) and eq. (17), we have:

$$cn^2\left(\xi\right)=1-w^2,\quad dn^2\left(\xi\right)=1-k^2w^2 \tag{25}$$

where $w=sn\left(\xi\right).$

Substituting eqs. (23)-(25) into eq. (22), we get:

$$6\omega A_1^2\left(b_1^2+2b_1b_2w+b_2^2w^2\right)+2p\varepsilon A_1^3\left(b_1+b_2w\right)+6r\varepsilon^3A_1^2b_2\left(1-k^2w^2-w^2+k^2w^4\right)=$$
$$=c_1A_1\left(b_1^3+3b_1^2b_2w+3b_1b_2^2w^2+b_2^3w^3\right)+c_2\left(b_1^4+4b_1^3b_2w+6b_1^2b_2^2w^2+4b_1b_2^3w^3+b_2^4w^4\right)$$

By comparing the coefficients of the same power of w in the previous equation, we get:

$$6\omega A_1^2b_1^2+2p\varepsilon A_1^3b_1+\varepsilon^3A_1^2b_2=c_1A_1b_1^3+c_2b_1^4 \tag{26}$$

$$12\omega A_1^2 b_1 b_2 + 2p\varepsilon A_1^3 b_2 = 3c_1 A_1 b_1^2 b_2 + 4c_2 b_1^3 b_2 \tag{27}$$

$$6\omega A_1^2 b_2^2 - 6\left(1+k^2\right) r\varepsilon^3 A_1^2 b_2^2 = 3c_1 A_1 b_2^2 b_1 + 6c_2 b_1^2 b_2^2 \tag{28}$$

$$c_1 A_1 b_2^3 + 4c_2 b_2^3 b_1 = 0 \tag{29}$$

$$6r\varepsilon^3 A_1^2 b_2^2 k^2 = c_2 b_2^4 \tag{30}$$

We can get infinitely many solutions for the system of eqs. (26)-(30). For example, solving the system of equations with the help of MAPLE we get the following set of solutions:

$$A_1 = 6p^{-1} r\varepsilon^2 \left(k^2 - 1\right), \quad b_1 = b_2 = 1, \quad \omega = r\varepsilon^3 \left(1 - 5k^2\right)$$

Thus, by eq. (18), we obtain the exact solution of eq. (2):

$$u(x,t) = \frac{6p^{-1} r\varepsilon^2 \left(k^2 - 1\right)}{1 + sn\left[\dfrac{\varepsilon\bar{\lambda}}{x^{\bar{\mu}-1}} + \dfrac{r\varepsilon^3 \omega\lambda\left(1 - 5k^2\right)}{t^{\mu-1}}\right]} \tag{31}$$

Then, we take $b_2 = 0.$ In this case, eq. (23) becomes:

$$u(\xi) = A + Bsn(\xi) \tag{32}$$

where $A = A_1/b_1, B = A_2/b_1.$

Substituting eq. (32) into eq. (22), we have:

$$3\omega\left(A^2 + 2ABw^2 + B^2 w^4\right) + p\varepsilon\left(A^3 + 3A^2 Bw^2 + 3AB^2 w^4 + B^3 w^6\right) +$$

$$+12r\varepsilon^3 B^2 w^2 \left(1 - k^2 w^2 - w^2 + k^2 w^4\right) = c_1 A + c_1 Bw^2 + c_2 \tag{33}$$

By comparing the coefficients of the same power of *w* in the previous equation, we get:

$$3\omega A^2 + p\varepsilon A^3 = c_1 A + c_2 \tag{34}$$

$$2\omega AB + p\varepsilon A^2 B + 4r\varepsilon^3 B^2 = c_1 B \tag{35}$$

$$\omega B^2 - p\varepsilon AB^2 - 4\left(1+k^2\right) r\varepsilon^3 B^2 = 0 \tag{36}$$

$$p\varepsilon B^3 + 12r\varepsilon^3 B^2 k^2 = 0 \tag{37}$$

Solving the system of equations, we get the following set of solutions:

$$A = 0, \quad B = 1, \quad \varepsilon^2 = -p\left(12rk^2\right)^{-1}, \quad c_1 = 12r\varepsilon^3, \quad c_2 = 0, \quad \omega = 4r\varepsilon^3\left(1 + k^2\right)$$

Hence, we obtain the exact solution of eq. (2) in the form:

$$u = sn^2\left[\frac{l\bar{\lambda}}{x^{\bar{\mu}-1}} + \frac{4r\varepsilon^3\omega\lambda\left(1+k^2\right)}{t^{\mu-1}}\right] \tag{38}$$

Next, we consider the case $q \neq 0$. In this case, we seek solutions of eq. (22) in the form:

$$u(\xi) = \frac{Msn(\xi)}{m_0 + m_1 sn(\xi) + m_2 cn(\xi)} \tag{39}$$

By using the same method as was used previously, we can obtain the following set of solutions:

$$M = m_0 = m_1 = m_2 = 1, \quad \varepsilon^2 = -q\left[3r\left(1-k^2\right)\right]^{-1}, \quad \omega = -r\varepsilon^3\left(2-k^2\right) \tag{40}$$

From eq. (40), we obtain the exact solution of eq. (2) in the form:

$$u(\xi) = \frac{sn(\xi)}{1 + sn(\xi) + cn(\xi)}, \quad \xi = \frac{\varepsilon\bar{\lambda}}{x^{\bar{\mu}-1}} + \frac{r\varepsilon^3\omega\lambda\left(2-k^2\right)}{t^{\mu-1}} \tag{41}$$

Remark 1. By eq. (14), as k approaches 0 from the right, eq. (31) becomes:

$$u(x,t) = \frac{-6p^{-1}r\varepsilon^2}{1 + \sin\left(\frac{\varepsilon\bar{\lambda}}{x^{\bar{\mu}-1}} + \frac{r\varepsilon^3\omega\lambda}{t^{\mu-1}}\right)}$$

and eq. (38) becomes:

$$u = \sin^2\left(\frac{l\bar{\lambda}}{x^{\bar{\mu}-1}} + \frac{4r\varepsilon^3\omega\lambda}{t^{\mu-1}}\right)$$

Remark 2. By eq. (15), as k approaches 1 from the right, eq. (41) becomes:

$$u(\xi) = \frac{\tanh(\xi)}{1 + \tanh(\xi) + \operatorname{sech}(\xi)}, \quad \xi = \frac{\varepsilon\bar{\lambda}}{x^{\bar{\mu}-1}} + \frac{r\varepsilon^3\omega\lambda}{t^{\mu-1}}$$

Conclusion

In this paper, we seek analytical solutions for a class of KdV-type equations involving Yang's fractal scaling-law derivative. The advent of Jacobi elliptic functions has led to the discovery of novel exact analytical solutions to this class of equations. The methodologies delineated in this article can be extrapolated to other non-linear fractal differential equations, *e.g.*, the fractal Richards equation [25], and the fractional order can be determined by the two-scale fractal dimensions [26, 27].

References

[1] Kato, T., On the Korteweg-de Vries Equation, *Manuscripta Math., 28* (1979), Jan., pp. 89-99
[2] Miles, J. W., The Korteweg-de Vries Equation: A Historical Essay, *Journal of Fluid Mechanics, 106* (1981), May, pp. 131-147

[3] Wazwaz, A.-M., The KdV Equation, *Handbook of Differential Equations: Evolutionary Equations, 4* (2008), 1, pp. 485-568
[4] Seadawy, A. R., New Exact Solutions for the KdV Equation with Higher Order Non-Linearity by Using the Variational Method, *Computers and Mathematics with Applications*, *10* (2011), 62, pp. 3741-3755
[5] Guan, Y. Z., *et al.*, Variational Formulations for a Coupled Fractal - Fractional KdV System, *Fractals*, *32* (2024), 24500543
[6] Shang, C. H., Yi, H. A., Solutions of the KdV - mKdV Equations Arising in Non - Linear Elastic Rods under Fractal Dimension *Thermal Science*, *28* (2024), 3A, pp. 2125-2133
[7] Sun, J. S., Fractal Solitary Waves of the (3+1)-Dimensional Fractal Modified KdV - Zakharov – Kuznetsov, *Thermal Science*, *28* (2024), 3A, 1967-1974
[8] He, J.-H., Wu, X. H., Exp - Function Method for Non-Linear Wave Equations, *Chaos Solitons & Fractals*, *30* (2006), 3, pp. 700-708
[9] Sajid, N., *et al.*, Implementation of the Exp - Function Approach for the Solution of KdV Equation with Dual Power Law Non-Linearity, *Computational & Applied Mathematics*, *41* (2022), 338
[10] Abdalla, M. Y. Y., *et al.*, Multiple Soliton Solutions of the Model mKdV with Time - Dependent Variable Coefficients, *Thermal Science*, *28* (2024), 6B, pp. 5029-5036
[11] He, J.-H, A Short Review on Review on Analytical Methods for a Fully Fourth-Order Nonlinear Integral Boundary Value Problem with Fractal Derivatives, *International Journal of Numerical Methods for Heat and Fluid Flow*, *11* (2020), 30, pp. 4933-4943
[12] Manddelbrot, B., How Long is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension, *Science*, 156 (1967), 3775, pp. 636-638
[13] Yang, X.-J., New Insight into the Fourier-Like and Darcy-Like Models in Porous Medium, *Thermal Science*, *24* (2020), 6A, pp. 3847-3858
[14] Yang, X.-J., The Scaling-Law Flows: An Attempt at Scaling-Law Vector Calculus, *Fractals*, 32 (2024), 2340126
[15] Grunau, D. W., *et al.*, Domain Growth, Wetting, and Scaling in Porous Media, *Physical Review Letters*, *71* (1993), 4198
[16] Hariti, Y. *et al.*, Scaling Law of Permeability and Porosity for Fluid Transport Phenomena in Porous PCM Media, *Journal of Applied and Computational Mechanics*, *7* (2021), 1, pp. 84-92
[17] Werner J. P., *et al.*, Crisis and Stochastic Resonance in Shinriki's Circuit, Physica D: Nonlinear Phenomena, *237* (2008), 6, pp. 859-865
[18] Paradisi, P., Allegrini, P., Scaling Law of Diffusivity Generated by a Noisy Telegraph Signal with Fractal Intermittency, *Chaos, Solitons and Fractals*, *81* (2015), Dec., pp. 451-462
[19] Rafe, A., Razavi, S. M. A., Scaling Law, Fractal Analysis and Rheological Characteristics of Physical Gels Cross-Linked with Sodium Trimetaphosphate, *Food Hydrocolloids*, *62* (2017), Jan., pp. 58-65
[20] Yuan, C., *et al.*, The Emergence of Scaling Law, Fractal Patterns and Small-World in Wireless Networks, *IEEE Access*, *5* (2017), Jan., pp. 3121-3130
[21] Meyer, K. R., Jacobi Elliptic Functions from a Dynamical Systems Point of View, *The American Mathematical Monthly*, *108* (2001), 8, pp. 729-737
[22] Chouikha, A. R., On Properties of Elliptic Jacobi Functions and Applications, *Journal of Nonlinear Mathematical Physics*, *12* (2005), 2, pp. 162-169
[23] Ain, Q. T., *et al.*, ABC Fractional Derivative for the Alcohol Drinking Model Using Two-Scale Fractal Dimension, *Complexity*, *2022* (2022), 8531858
[24] Zhang, Y. R., *et al.*, Fast and Accurate Population Forecasting with Two-Scale Fractal Population Dynamics and its Application to Population Economics, *Fractals*, *32* (2024), 2450082
[25] He, J.-H., *et al.*, A Fractal-Based Approach to the Mechanical Properties of Recycled Aggregate Concretes, *Facta Universitatis, Series: Mechanical Engineering*, *22* (2024), 2, pp. 329-342
[26] He, J.-H., Liu, C., Fractal Dimensions of a Porous Concrete and Its Effect on the Concrete's Strength, *Facta Universitatis Series: Mechanical Engineering*, *21* (2023), 1, pp. 137-150
[27] Liu, Y. P., *et al.*, Clover-Inspired Fractal Architectures: Innovations in Flexible Folding Skins for Sustainable Buildings, *Fractals*, *33* (2025), 25500410

Paper submitted: October 8, 2024
Paper revised: May 2, 2025
Paper accepted: May 5, 2025