THE HOMOTOPY PERTURBATION METHOD FOR THE DOUBLE PERIOD SOLUTION OF LOCAL FRACTIONAL KORTEWEG-DE VRIES EQUATION

by

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The Korteweg-de Vries equation is a fundamental equation for the study of shallow water waves and plays a crucial role in fluid physics and applied mathematics. The objective of this study is to implement the homotopy perturbation method to solve a class of Korteweg-de Vries equations with bi-periodic numerical solutions. The method is predicated on the selection of appropriate iterative operations, thereby ensuring the numerical solution of the equation is obtained in the form of the trigonometric series.

Keywords: *Korteweg-de Vries equation, fractional systems, period solution, local fractional derivative, homotopy perturbation method*

Introduction

The Korteweg-de Vries (KdV) equation was first introduced by Korteweg and de Vries in [1] as a means of simulating shallow water wave equations of small and finite amplitudes. Following the discovery of the solution by Zabusky and Kruskal [2], a significant interest has been expressed by the physical and mathematical communities in studying the solution to the KdV equation. A plethora of analytical and numerical methodologies for solving the equation are provided. For instance, the pseudo-spectrum method [3], the heat balance integral method [4], and the finite element technique [5] are notable examples. It is noteworthy that Miura [6] provides a comprehensive review of the applications of the KdV equation.

In contrast to integral calculus and other fractional calculus, local fractional KdV equations have the capacity to describe numerous anomalies in theoretical and engineering contexts. In recent years, there has been an increasing interest among researchers in the local fractional KdV equations [7-10]. In this paper, we employ the homotopy perturbation method [11-18] to elucidate the problem of bi-periodic solutions of KdV equations with local fractional derivatives.

The basic concepts of local fractional calculus

The local fractional calculus is a field of study that focuses on the calculus of functions on Cantor fractals. The local fractional calculus, also referred to as fractal calculus, constitutes a pivotal branch of pure mathematics. In this section, we provide a concise overview

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of the fundamental definitions of local fractional calculus, including continuity, derivative, and integral.

Definition 1 [19, 20] In Cantor fractal space, if the function meets $K(t)$:

$$\left|K(t)-K(t_0)\right| < \varepsilon^\alpha \tag{1}$$

with $|t - t_0| < \delta$, for ε, $\delta > 0$ and ε, $\delta \in R$, then $K(t)$ is defined as the fractional continuous at $t = t_0$ and it is expressed as:

$$\lim_{t\to t_0} K(t) = K(t_0)$$

Definition 2 [19, 20] If the function $K(t)$ is satisfied the condition (1) for any $t \in (c, d)$, it is defined as the continuous on the set (c, d), denoted by:

$$K(t) \in C_\alpha(c,d) \tag{2}$$

Definition 3 [19, 20] If $K(t) \in C_\alpha(c, d)$, the fractional derivative of $K(t)$ of order α at $t = t_0$ is defined by:

$$\mathrm{D}_t^{(\alpha)}K(t_0) = K^{(\alpha)}(t_0) = \frac{\mathrm{d}^\alpha K(t)}{\mathrm{d}t^\alpha}\Big|_{t=t_0} = \lim_{t\to t_0}\frac{\Delta^\alpha[K(t)-K(t_0)]}{(t-t_0)^\alpha} \tag{3}$$

where $\Delta^\alpha[K(t)-K(t_0)] \cong \Gamma(1+\alpha)\Delta[K(t)-K(t_0)]$.

Definition 4 [19, 20] Let $K(t) \in C_\alpha(c, d)$, the fractional integral of $K(t)$ of order αin the interval $[c, d]$ is defined by:

$${}_c\mathrm{I}_d^{(\alpha)}\mathrm{K}(t) = \frac{1}{\Gamma(1+\alpha)}\int_c^d \mathrm{K}(t)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}\mathrm{K}(t_j)(\Delta t_j)^\alpha \tag{4}$$

where $\Delta t_j = t_{j+1}$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j\ldots\}$, and $[t_j, t_{j+1}]$, $j = 0,\ldots N-1$, $t_0 = c$, $t_N = d$, is a partition of the interval $[c, d]$.

The homotopy perturbation method for the double periods solution of local fractional Korteweg-de Vries equation

In this paper, the bi-periodic solution of the local fractional KdV equation is studied by the homotopy perturbation method [21-23].

The local fractional KdV equation for fractal wave on shallow water surface can generally be written as [24]:

$$u_x^{(3\alpha)} = \gamma u_t + \beta u u_x^{(\alpha)}, \quad u(0,0) = A, \quad \frac{\partial^\alpha u(0,0)}{\partial t^\alpha} = 0 \tag{5}$$

where $u = u(x, t)$ is the fractal wave function of single spatial variable x and time, t, A – the constant, and α – the fractal dimension. Moreover, the parameters $\gamma(\gamma < 0)$ and β characterize the fractal waves on the surface of shallow water.

Obviously, we can choose $u = A\cos_\alpha(\varpi_1 x + \omega_2 t)^\alpha$ as the initial solution of (5), which is a modified function of $u = A\cos\left(\varpi_1 x + \omega_2 t\right)$ in the local fractional sense.

We construct the following homotopy equation [17, 18]:

$$u_x^{(3\alpha)} - \gamma u_t + \frac{\varpi_1^3 + \gamma\omega_2}{\omega_1}u_x^{(\alpha)} = p\left[\frac{\varpi_1^3 + \gamma\omega_2}{\omega_1}u_x^{(\alpha)} + \beta u u_x^{(\alpha)}\right] \tag{6}$$

where ϖ is the homotopy parameter. When $p=0$, eq. (6) is a linearized equation. And when $p = 1$, eq. (6) becomes eq. (5).

According to the requirements of the homotopy perturbation method [17, 18], we assume:

$$u=u_0+pu_1+p^2u_2+... \tag{7}$$

where $u_0=A\cos_\alpha(\varpi_1 x+\omega_2 t)^\alpha$.

By virtue of eq. (7), eq. (6) can be decomposed into a series of linear equations:

$$\begin{gathered}\sum_{n=0}^{\infty}p^n u_{nx}^{(3\alpha)}-\gamma\sum_{n=0}^{\infty}p^n u_{nt}^{(\alpha)}+\frac{\varpi_1^3+\gamma\omega_2}{\omega_1}\sum_{n=0}^{\infty}p^n u_{nx}^{(\alpha)}=\\ =p\left[\frac{\varpi_1^3+\gamma\omega_2}{\omega_1}\sum_{n=0}^{\infty}p^n u_{nx}^{(\alpha)}+\beta\sum_{n=0}^{\infty}p^n u_n\sum_{n=0}^{\infty}p^n u_{nx}^{(\alpha)}\right]\\ u_0(0,0)=A,\quad \frac{\partial^\alpha u_0}{\partial t^\alpha}(0,0)=0,\quad u_n(0,0)=0,\quad \frac{\partial^\alpha u_n}{\partial t^\alpha}(0,0)=0\left(n\ge 1\right)\end{gathered} \tag{8}$$

The first two linear differential equations of eq. (8) are selected:

$$u_{0x}^{(3\alpha)}-\gamma u_{0t}+\frac{\varpi_1^3+\gamma\omega_2}{\omega_2}u_{0x}^{(\alpha)}=0 \tag{9}$$

and

$$u_{1x}^{(3\alpha)}-\gamma u_{1t}+\frac{\varpi_1^3+\gamma\omega_2}{\omega_2}u_{1x}^{(\alpha)}=\frac{\varpi_1^3+\gamma\omega_2}{\omega_2}u_{0x}^{(\alpha)}+\beta u_0 u_{0x}^{(\alpha)} \tag{10}$$

By solving eq. (9), we can get:

$$u_0(x,t)=A\cos_\alpha(\varpi_1 x+\omega_2 t)^\alpha \tag{11}$$

Substituting eq. (11) into eq. (10), eq. (10) is deformed:

$$\begin{gathered}u_{1x}^{(3\alpha)}-\gamma u_{1t}+\frac{\varpi_1^3+\gamma\omega_2}{\omega_2}u_{1x}^{(\alpha)}=\frac{\varpi_1^3+\gamma\omega_2}{\omega_2}A\varpi_1\cos_\alpha(\varpi_1 x+\omega_2 t)^\alpha-\\ -\beta A^2\varpi_1\cos_\alpha(\varpi_1 x+\omega_2 t)^\alpha\sin_\alpha(\varpi_1 x+\omega_2 t)^\alpha\end{gathered} \tag{12}$$

In order to derive a periodic solution to u_1, the coefficient of $\cos_\alpha(\varpi_1 x+\omega_2 t)^\alpha$ must be zero, that is:

$$\frac{\varpi_1^3+\gamma\omega_2}{\omega_2}A\varpi_1=0 \tag{13}$$

Thence:

$$\varpi_1^3+\gamma\omega_2=0 \tag{14}$$

Simplify eq. (12):

$$u_{1x}^{(3\alpha)} - \gamma u_{1t} = -\frac{\beta A^2 \varpi_1}{2}\sin_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha \tag{15}$$

Let:

$$u_1(x,t) = C_1\cos_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha + C_2\cos_\alpha(\varpi_1 x + \omega_2 t)^\alpha + C_3 \tag{16}$$

Substituting eq. (16) into eq. (15), we can get:

$$\left[C_1\left(2\varpi_1\right)^3 + C_1 2\varpi_2\gamma\right]\sin_\alpha\left(2\varpi_1 x + 2\omega_2 t\right)^\alpha = -\frac{\beta A^2\varpi_1}{2}\sin_\alpha\left(2\varpi_1 x + 2\omega_2 t\right)^\alpha \tag{17}$$

According to the coefficient of the same term of $\sin_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha$ of eq. (17), we can get:

$$C_1 = -\frac{\beta A^2}{12\varpi_1^2} \tag{18}$$

By virtue of (16) and (18), we can get the solution of eq. (15):

$$u_1(x,t) = -\frac{\beta A^2}{12\varpi_1^2}\cos_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha + C_2\cos_\alpha(\varpi_1 x + \omega_2 t)^\alpha + C_3 \tag{19}$$

By virtue of eqs. (11) and (19), the third linear differential equation of eq. (8) is selected and simplified:

$$u_{2x}^{(3\alpha)} - \gamma u_{2t} = \beta u_0 u_{1x} + \beta u_{0x} u_1 = \frac{\beta^2 A^3}{8\varpi_1}\sin_\alpha\left(3\varpi_1 x + 3\omega_2 t\right)^\alpha -$$

$$-\left(\frac{AC_2\varpi_1 + A\varpi_1 C_2}{2}\right)\sin_\alpha\left(2\varpi_1 x + 2\omega_2 t\right)^\alpha - \left(-\frac{\beta^2 A^3}{24\varpi_1} + A\varpi_1 C_3\right)\sin_\alpha\left(\varpi_1 x + \omega_2 t\right)^\alpha \tag{20}$$

In order to be a periodic solution of u_2, the coefficient of $\sin_\alpha\left(\varpi_1 x + \omega_2 t\right)^\alpha$ must be zero.

Thence:

$$C_3 = \frac{\beta A^2}{24\varpi_1^2} \tag{21}$$

By virtue of eqs. (19) and (21), and $u_1(0, 0) = 0$, $[\partial^\alpha u_1(0, 0)]/(\partial t^\alpha) = 0$, we can get the solution of eq. (15):

$$u_1(x,t) = -\frac{\beta A^2}{12\varpi_1^2}\cos_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha + \frac{\beta A^2}{24\varpi_1^2}\cos_\alpha(\varpi_1 x + \omega_2 t)^\alpha + \frac{\beta A^2}{24\varpi_1^2} \tag{22}$$

The first approximate solution of eq. (5) can be derived:

$$u(x,t) = u_0 + u_1 = -\frac{\beta A^2}{12\varpi_1^2}\cos_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha + \left(A + \frac{\beta A^2}{24\varpi_1^2}\right)\cos(\varpi_1 x + \omega_2 t)^\alpha + \frac{\beta A^2}{24\varpi_1^2} \tag{23}$$

where ϖ_1 and ϖ_2 satisfy eq. (14).

Similarly, as for:

$$u_{2x}^{(3\alpha)} - \gamma u_{2t} = \beta u_0 u_{1x} + \beta u_{0x} u_1 \tag{24}$$

By virtue of Eq. (11), Eq. (22), and Eq. (24), we can derive:

$$u_{2x}^{(3\alpha)} - \gamma u_{2t}^{(\alpha)} = \frac{\beta^2 A^3}{8\varpi_1}\sin_\alpha(3\varpi_1 x + 3\omega_2 t)^\alpha - \frac{\beta^2 A^3}{24\varpi_1}\sin_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha \tag{25}$$

Let:

$$u_2\left(x,t\right) = C_1\cos_\alpha(3\varpi_1 x + 3\omega_2 t)^\alpha + C_2\cos_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha + C_3\cos_\alpha(\varpi_1 x + \omega_2 t)^\alpha + C_4 \tag{26}$$

Substituting eq. (26) into eq. (25), we can get the following equation:

$$\begin{aligned} u_{2x}^{(3\alpha)} - \gamma u_{2t} &= \left[\left(3\varpi_1\right)^3 C_1 + 3\varpi_2 C_1\gamma\right]\sin_\alpha\left(3\varpi_1 x + 3\omega_2 t\right)^\alpha + \\ &+ \left[\left(2\varpi_1\right)^3 C_2 + 2\varpi_2 C_2\gamma\right]\sin_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha + \left[\varpi_1{}^3 C_3 + \varpi_2 C_3\gamma\right]\sin_\alpha(\varpi_1 x + \omega_2 t)^\alpha = \\ &= \frac{\beta^2 A^3}{8\varpi_1}\sin_\alpha(3\varpi_1 x + 3\omega_2 t)^\alpha - \frac{\beta^2 A^3}{24\varpi_1}\sin_\alpha(2\varpi_1 x + 2\omega_2 t)^\alpha \end{aligned} \tag{27}$$

Comparing the coefficients of the same type of $\sin_\alpha(n\varpi_1 x + n\omega_2 t)^\alpha$ of eq. (27), we can get:

$$\left(3\varpi_1\right)^3 C_1 + 3\varpi_2 C_1\gamma = \frac{\beta^2 A^3}{8\varpi_1}, \quad \left(2\varpi_1\right)^3 C_2 + 2\varpi_2 C_2\gamma = -\frac{\beta^2 A^3}{24\varpi_1} \tag{28}$$

This is:

$$C_1 = \frac{\beta^2 A^3}{8\cdot 24\varpi_1^4}, \quad C_2 = \frac{-\beta^2 A^3}{6\cdot 24\varpi_1^4} \tag{29}$$

Thence:

$$\begin{aligned} u_2(x,t) &= \frac{\beta^2 A^3}{8\times 24\varpi_1^4}\cos_\alpha\left(3\varpi_1 x + 3\omega_2 t\right)^\alpha - \frac{\beta^2 A^3}{6\times 24\varpi_1^4}\cos_\alpha\left(2\varpi_1 x + 2\omega_2 t\right)^\alpha + \\ &+ C_3\cos_\alpha\left(\varpi_1 x + \omega_2 t\right)^\alpha + C_4 \end{aligned} \tag{30}$$

Similarly, as for:

$$u_{3x}^{(3\alpha)} - \gamma u_{3t} = \beta u_0 u_{2x} + \beta u_1 u_{1x} + \beta u_{0x} u_2 \tag{31}$$

By virtue of eqs. (11), (22), (30), and (31), we can get:

$$u_{3x}^{(3\alpha)}-\gamma u_{3t}=\beta u_0u_{2x}+\beta u_1u_{1x}+\beta u_{0x}u_2$$

$$=\beta\left[-\frac{\beta^2A^4}{2\cdot 8\cdot 24\varpi_1^3}-\frac{\beta^2A^4}{128\varpi_1^3}-\frac{\beta^2A^4}{12\cdot 6\cdot 2\varpi_1^3}\right]\sin_\alpha 4\Delta+$$

$$+\beta\left[\frac{\beta^2A^4}{2\cdot 6\cdot 24\varpi_1^3}+\frac{\beta^2A^4}{6\cdot 24\varpi_1^3}+\frac{\beta^2A^4}{24\cdot 24\varpi_1^3}+\frac{\beta^2A^4}{24\cdot 6\cdot 2\varpi_1^3}\right]\sin_\alpha 3\Delta+$$

$$+\beta\left[\frac{\beta^2A^4}{2\cdot 8\cdot 24\varpi_1^3}-\frac{A}{2}\varpi_1C_3-\frac{\beta^2A^4}{128\varpi_1^3}-\frac{A}{2}\varpi_1C_3-\frac{\beta^2A^4}{24\cdot 24\cdot 2\varpi_1^3}+\frac{\beta^2A^4}{24\cdot 6\varpi_1^3}\right]\sin_\alpha 2\Delta+$$

$$+\beta\left[-\frac{\beta^2A^4}{2\cdot 6\cdot 24\varpi_1^3}-A\varpi_1C_4+\frac{\beta^2A^4}{6\cdot 24\varpi_1^3}-\frac{\beta^2A^4}{24\cdot 24\varpi_1^3}+\frac{\beta^2A^4}{24\cdot 6\cdot 2\varpi_1^3}-\frac{\beta^2A^4}{24\cdot 24\varpi_1^3}\right]\sin_\alpha\Delta \quad (32)$$

where $\Delta=(\varpi_1x+\omega_2t)^\alpha$.

In order to be a periodic solution of u_3, the coefficient of $\sin_\alpha(\varpi_1x+\omega_2t)^\alpha$ must be zero, that is:

$$C_4=\frac{\beta^2A^3}{12\cdot 24\varpi_1^4} \quad (33)$$

Due to eqs. (30) and (33), and $u_2(0,\,0)=0$, $[\partial^\alpha u_2(0,\,0)]/(\partial t^\alpha)=0$, we get C_3 of eq. (30):

$$C_3=-\frac{\beta^2A^3}{24\cdot 24\varpi_1^4} \quad (34)$$

This is:

$$u_2(x,t)=\frac{\beta^2A^3}{192\varpi_1^4}\cos_\alpha(3\varpi_1x+3\omega_2t)^\alpha-\frac{\beta^2A^3}{144\varpi_1^4}\cos_\alpha(2\varpi_1x+2\omega_2t)^\alpha-$$

$$-\frac{\beta^2A^3}{576\varpi_1^4}\cos_\alpha(\varpi_1x+\omega_2t)^\alpha+\frac{\beta^2A^3}{288\varpi_1^4} \quad (35)$$

The second approximate solution of eq. (5) can be derived:

$$u(t)=u_0+u_1+u_2=\frac{\beta^2A^3}{192\varpi_1^4}\cos_\alpha(3\varpi_1x+3\omega_2t)^\alpha+\left(-\frac{\beta A^2}{12\varpi_1^2}-\frac{\beta^2A^3}{144\varpi_1^4}\right)\cos_\alpha(2\varpi_1x+2\omega_2t)^\alpha+$$

$$+\left(A+\frac{\beta A^2}{24\varpi_1^2}-\frac{\beta^2A^3}{576\varpi_1^4}\right)\cos_\alpha(\varpi_1x+\omega_2t)^\alpha+\left(\frac{\beta A^2}{24\varpi_1^2}+\frac{\beta^2A^3}{288\varpi_1^4}\right) \quad (36)$$

As for:

$$u_{3x}^{(3\alpha)}-\gamma u_{3t}=\beta u_0u_{2x}+\beta u_{1x}u_1+\beta u_{0x}u_2 \quad (37)$$

by virtue of eqs. (8), (11), (22), (35), and (37), we can derive:

$$u_{3x}^{(3\alpha)}-\gamma u_{3t}=\beta u_0u_{2x}+\beta u_{1x}u_1+\beta u_{0x}u_2=-\frac{5}{288\varpi_1^3}\beta^3A^4\sin_\alpha\left(4\varpi_1x+4\omega_2t\right)^\alpha+$$

$$+\frac{\beta^3A^4}{64\varpi_1^3}\sin_\alpha\left(3\varpi_1x+3\omega_2t\right)^\alpha+\frac{\beta^3A^4}{384\varpi_1^4}\sin_\alpha\left(2\varpi_1x+2\omega_2t\right)^\alpha \tag{38}$$

and

$$u_3\left(0,0\right)=0,\quad \frac{\partial^\alpha u_3\left(0,0\right)}{\partial t^\alpha}=0 \tag{39}$$

Let:

$$u_3(x,t)=C_1\cos_\alpha\left(4\varpi_1x+4\omega_2t\right)^\alpha+C_2\cos_\alpha\left(3\varpi_1x+3\omega_2t\right)^\alpha+$$

$$+C_3\cos_\alpha\left(2\varpi_1x+2\omega_2t\right)^\alpha+C_4\cos_\alpha\left(\varpi_1x+\omega_2t\right)^\alpha+C_5 \tag{40}$$

Substituting eq. (40) into eq. (38), we can get the following equation:

$$u_{3x}^{(3\alpha)}-\gamma u_{3t}^{(\alpha)}=$$

$$=\left[-\frac{\beta^3A^4}{64\varpi_1^3}\frac{1}{2}-\frac{\beta^3A^4}{72\varpi_1^3}\frac{1}{2}-\frac{1}{8}\frac{\beta^3A^4}{24\varpi_1^3}\frac{1}{2}\right]\sin_\alpha\left(4\varpi_1x+4\omega_2t\right)^\alpha+$$

$$+\left[\frac{7}{24}\frac{\beta^3A^4}{24\varpi_1^3}-\frac{1}{24}\frac{\beta^3A^4}{6\varpi_1^3}\frac{1}{2}\right]\sin_\alpha\left(3\varpi_1x+3\omega_2t\right)^\alpha+$$

$$+\left[-\frac{3\beta^3A^4}{64\varpi_1^3}\frac{1}{2}-\frac{\beta^3A^4}{24^2\varpi_1^3}\frac{1}{2}+\frac{1}{24}\frac{\beta^3A^4}{6\varpi_1^3}+\frac{1}{8}\frac{\beta^3A^4}{24\varpi_1^3}\frac{1}{2}-\frac{\beta^3A^4}{24\left(24\varpi_1^3\right)}\right]\sin_\alpha\left(2\varpi_1x+2\omega_2t\right)^\alpha \tag{41}$$

According to eqs. (38) and (41), comparing the coefficients of the same type of $\sin_\alpha(n\varpi_1x+n\omega_2t)^\alpha$, we can get:

$$C_1=-\frac{\beta^3A^4}{288\cdot12\varpi_1^3},\quad C_2=\frac{\beta^3A^4}{64\cdot24\varpi_1^6},\quad C_3=\frac{\beta^3A^4}{384\cdot6\varpi_1^6} \tag{42}$$

Thence:

$$u_3(x,t)=-\frac{\beta^3A^4}{288\cdot12\varpi_1^3}\cos_\alpha\left(4\varpi_1x+4\omega_2t\right)^\alpha+\frac{\beta^3A^4}{64\cdot24\varpi_1^6}\cos_\alpha\left(3\varpi_1x+3\omega_2t\right)^\alpha+$$

$$+\frac{\beta^3A^4}{384\cdot6\varpi_1^6}\cos_\alpha\left(2\varpi_1x+2\omega_2t\right)^\alpha+C_4\cos_\alpha\left(\varpi_1x+\omega_2t\right)^\alpha+C_5$$

By virtue of (7) and (8), we can get:

$$u_{4x}^{(3\alpha)}-\gamma u_{4t}=\beta u_0u_{3x}+\beta u_1u_{2x}+\beta u_2u_{1x}+\beta u_3u_{0x} \tag{43}$$

Similarly, in order to be a periodic solution of u_4, the coefficient of $\sin_\alpha(\varpi_1 x+\omega_2 t)^\alpha$ must be zero, that is:

$$-\frac{\beta^3A^5\sin\Delta}{192\cdot 6\cdot 2\varpi_1^5}+\frac{\beta^3A^5\sin\Delta}{2\cdot 12\cdot 8\cdot 8\varpi_1^5}+\frac{\beta^3A^5\sin\Delta}{12\cdot 24\cdot 24\cdot 2\varpi_1^5}+\frac{\beta^3A^5\sin\Delta}{24\cdot 24\cdot 3\cdot 2\varpi_1^5}+\frac{\beta^3A^5\sin\Delta}{24\cdot 24\cdot 24\varpi_1^5}-$$

$$-\frac{\beta^3A^5\sin\Delta}{8\cdot 24\cdot 6\cdot 2\varpi_1^5}-\frac{\beta^3A^5\sin\Delta}{24\cdot 6\cdot 24\cdot 2\varpi_1^5}-\frac{\beta^3A^5\sin\Delta}{24\cdot 24\cdot 6\cdot 2\varpi_1^5}-\frac{\beta^3A^5\sin\Delta}{12\cdot 24\cdot 24\varpi_1^5}-AC_5\varpi_1\sin\Delta=0 \quad (44)$$

Thence:

$$C_5=-\frac{\beta^3A^4}{4608\varpi_1^6} \tag{45}$$

Due to eqs. (40), (45), and $u_3(0, 0) = 0$, $[\partial^\alpha u_3(0, 0)]/(\partial t^\alpha) = 0$, we can get C_4 of eq. (40):

$$C_4=-\frac{\beta^3A^4}{1728\varpi_1^6} \tag{46}$$

This is:

$$u_3(x,t)=-\frac{\beta^3A^4}{3456\varpi_1^6}\cos_\alpha\left(4\varpi_1x+4\omega_2t\right)^\alpha+\frac{\beta^3A^4}{1536\varpi_1^6}\cos_\alpha\left(3\varpi_1x+3\omega_2t\right)^\alpha+$$

$$+\frac{\beta^3A^4}{2304\varpi_1^6}\cos_\alpha\left(2\varpi_1x+2\omega_2t\right)^\alpha-\frac{\beta^3A^4}{1728\varpi_1^6}\cos_\alpha\left(\varpi_1x+\omega_2t\right)^\alpha-\frac{\beta^3A^4}{4608\varpi_1^6} \tag{47}$$

Then, the third approximate solution of eq. (5) can be derived:

$$u(t)=u_0+u_1+u_2+u_3=-\frac{\beta^3A^4}{3456\varpi_1^6}\cos_\alpha(4\varpi_1x+4\omega_2t)^\alpha+$$

$$+\left[\frac{\beta^2A^3}{192\varpi_1^4}+\frac{\beta^3A^4}{1536\varpi_1^6}\right]\cos_\alpha(3\varpi_1x+3\omega_2t)^\alpha+\left(\frac{\beta^3A^4}{2304\varpi_1^6}-\frac{\beta^2A^3}{144\varpi_1^4}-\frac{\beta A^2}{12\varpi_1^2}\right)$$

$$\cos_\alpha(2\varpi_1x+2\omega_2t)^\alpha+\left[A+\frac{\beta A^2}{24\varpi_1^2}-\frac{\beta^2A^3}{576\varpi_1^4}-\frac{\beta^3A^4}{1728\varpi_1^6}\right]\cos_\alpha(\varpi_1x+\omega_2t)^\alpha+$$

$$+\left(\frac{\beta A^2}{24\varpi_1^2}+\frac{\beta^2A^3}{288\varpi_1^4}-\frac{\beta^3A^4}{4608\varpi_1^6}\right) \tag{48}$$

Conclusion

In this paper, the method has been successfully applied to solve the KdV equation, which is effective in deriving period numerical solutions with specific initial conditions. The numerical periodic solution can be obtained without calculating restrictive assumptions and transformations, as is the case in other conventional methods. The findings of these studies are regarded as contributing to an enhanced physical understanding of non-linear wave dynamics governed by periodic KdV equations.

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