STOCHASTIC BIFURCATION AND BISTABLE BEHAVIORS IN A MODIFIED FRACTIONAL RAYLEIGH SYSTEM DRIVEN BY RECYCLING NOISE

by

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The present study investigates the stochastic response and bifurcation behavior of system amplitude in a fractional and generalized bi-stable Van der Pol system driven by Gaussian colored noise. Firstly, the principle of minimal mean square error and the generalized harmonic balance technique were employed to demonstrate that the fractional derivative is equivalent to a linear combination of damping and restoring forces. Consequently, the original system was simplified to an equivalent integer order Van der Pol system. Secondly, the system amplitude's stationary probability density function is acquired by stochastic averaging. According to the principles of singularity theory, the critical parametric conditions for the stochastic P-bifurcation of system amplitude can be determined. A qualitative analysis of the stationary probability density function curves of amplitude is finally conducted in each area, with the transition set curves serving as a dividing point. The congruence between the analytical outcomes and the numerical results derived from the Monte-Carlo simulation substantiates the theoretical analysis in this paper. The methodology employed and the results obtained in this paper can enhance the design of the fractional-order controller to regulate the response of the system.

Keywords: *stochastic P-bifurcation, Gaussian colored noise, fractional damping, critical parametric conditions, Monte-Carlo simulation*

Introduction

Fractional calculus is a generalization of integer-order calculus. The integer-order derivative is incapable of expressing the memory characteristics of viscoelastic substances. The definition of the fractional derivative incorporates convolution, which can express a memory effect and demonstrate a cumulative effect over time. Consequently, the fractional derivative emerges as a more suitable mathematical instrument for the description of memory characteristics [1-4]. It has become a powerful mathematical instrument for research in fields such as anomalous diffusion, non-Newtonian fluid mechanics, viscoelastic mechanics, and soft matter

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physics. A comparison of the integer-order calculus and the fractional derivative reveals that the latter can describe various reaction processes with greater accuracy [5-7]. Consequently, it is necessary and significant to study the mechanical characteristics and the fractional order parametric influences on such systems.

Recently, many scholars have studied the dynamic behavior of non-linear multi-stable systems under different noise excitations and achieved fruitful results. Liu *et al.* [8] studied the response of a strongly non-linear vibro-impact system with Coulomb friction excited by real noise, and analyzed the P-bifurcation by a qualitative change of the friction amplitude and the restitution coefficient on the stationary probability distribution. Some researchers [9-11] studied the Van der Pol-Duffing oscillators under Levy noise, color noise, combined harmonic and random noise, respectively. The stochastic P-bifurcation behaviors of the noise oscillators are discussed by analyzing changes in the system stationary probability density function (PDF). The analytical results of the bimodal stationary PDF are obtained, showing that the system parameters and noise intensity can each induce stochastic P-bifurcation of the systems. Hao and Wu [12-14] investigated the tri-stable stochastic P-bifurcation in a generalized Duffing-Van der Pol oscillator under additive Gaussian white noise, multiplicative colored noise, combined additive and multiplicative Gaussian white noise, respectively. They obtained an analytical expression of the system stationary PDF of amplitude and analyzed the influences of noise intensity and system parameters on the system stochastic P-bifurcation. Chen amd Zhu [15] studied the response of a Duffing system with fractional damping under the combined white noise and harmonic excitations, and showed that variation of the fractional derivative order can arouse the system stochastic P-bifurcation. Huang and Jin [16] discussed the response and the stationary PDF of a single-degree-of-freedom strongly non-linear system under Gaussian white noise excitation. Li *et al.* [17] studied the bi-stable stochastic P-bifurcation behavior of a Van der Pol-Duffing system with the fractional derivative under additive and multiplicative colored noise excitations and found that changes in the linear damping coefficient, the fractional derivative order and the noise intensity can each lead to stochastic P-bifurcation in the system. Liu *et al.* [18] investigated a Duffing oscillator system with fractional damping under combined harmonic and Poisson white noise parametric excitation, and then the asymptotic Lyapunov stability with probability of the original system is analyzed based on the largest Lyapunov exponent [1]. Chen *et al.* [19] studied the primary resonance response of a Van der Pol system under fractional-order delayed negative feedback and forced excitation, and obtained the approximate analytical solution based on the averaging method. Chen *et al.* [20] proposed a stochastic averaging technique which can be used to study the randomly excited strongly non-linear system with delayed feedback fractional-order proportional-derivative controller, and obtained the stationary PDF of the system. A number of non-linear vibration systems [21-29], including those with extensive applications in energy harvesting and control, have been the subject of extensive study. These include the six degrees-of-freedom system and the 3-DOF auto-parametric system.

Due to complexity of the fractional derivative, the parametric vibration characteristics of the fractional system can only be analyzed qualitatively, while the critical conditions of the parametric influences can not be obtained. In practice, the critical conditions of the parametric influences play a vital role for the analysis and design of the fractional order systems. Additionally, the stochastic P-bifurcation of bi-stability for the generalized Van der Pol system with the fractional damping has not been reported in the open literature. In this paper, taking a generalized Van der Pol system with a fractional damping excited by multiplicative Gaussian white noise excitation as the example, non-linear vibration of this kind of fractional order systems are studied through the fractional derivative. The transition set curves and critical parameter conditions for the system stochastic P-bifurcation are obtained by the singularity method. The types of the system stationary PDF curves in each area of the parameter plane are analyzed. We also compare the numerical results from Monte-Carlo simulation with analytical solutions obtained by stochastic averaging. The comparison shows that the numerical results are in good agreement with the analytical solutions, verifying our theoretical analysis.

Derivation of the equivalent system

The initial condition of the Riemann-Liouville derivative has no physical meaning, while the initial condition of the system described by the Caputo derivative has not only clear physical meaning but also forms the same initial condition with the integer-ODE. Therefore, in this paper we adopt the following Caputo fractional derivative:

$$\begin{gathered} {}_a^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_a^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\,\mathrm{d}u, \quad m < p < m+1 \\ {}_a^C\mathrm{D}^p[x(t)] = x^{(m)}(t), \quad p = m \end{gathered} \tag{1}$$

where $m \in N$, $t \in [a,b]$, $x^{(m)}(t)$ is the m-order derivative of $x(t)$ and $\Gamma(m)$ is the Gamma function which satisfies $\Gamma(m+1) = m\Gamma(m)$.

For a given physical system, the initial moment of oscillators is $t = 0$ and the Caputo derivative is usually expressed:

$${}_0^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\,\mathrm{d}u \tag{2}$$

where $m-1 < p \le m, m \in N$.

In this paper, we study the generalized Van der Pol system with the fractional damping excited by Gaussian colored noise excitation:

$$\ddot{x}(t) - u[-\varepsilon + \alpha_1\dot{x}^2(t) - \alpha_2\dot{x}^4(t)]{}_0^C\mathrm{D}^p[x(t)] + w^2x(t) = [\alpha + (1-\alpha)x(t)]\eta(t) \tag{3}$$

Which is equivalent to the following three dimensional systems:

$$\begin{gathered} {}_0^C\mathrm{D}^p[x(t)] = y(t), \quad {}_0^C\mathrm{D}^{1-p}[y(t)] = z(t) \\ \dot{z}(t) - \mu[-\varepsilon + \alpha_1x^2(t) - \alpha_2x^4(t)]y(t) + w^2x(t) = [\alpha + (1-\alpha)x(t)]\eta(t) \end{gathered} \tag{4}$$

where $x(t)$ is the displacement of the system, ε – the linear damping coefficient, α_1 and α_2 – the non-linear damping coefficients of the system, and w – the system natural frequency. The ${}_0^C\mathrm{D}^p[x(t)]$ is the p ($0 \le p \le 1$) order Caputo derivative of $x(t)$, which is defined by eq. (2). The $\xi(t)$ denotes the master noise, $k\,\xi(t-\tau)$ signifies the recycled noise (secondary noise), k – the relative strength, and τ – the recycling lag. In addition $\xi(t)$ is a Gaussian white noise, whose statistical properties are determined by:

$$E[\xi(t)\xi(t+t')] = 2D\delta(t'), \quad E[\xi(t)] = 0$$

Setting $\eta(t)=\xi(t)+k\xi(t-\tau)$, then it is called the recycling noise, which satisfies:

$$
\begin{gathered}
E[\eta(t)]=0 \\
E[\eta(t)\eta(t+t')]=2D[(1+k^2)\delta(t')+k\delta(t'+\tau)+k\delta(t'-\tau)] \\
S(w)=2D[1+k^2+2k\cos(w\tau)]
\end{gathered}
\tag{5}
$$

According to (3), it is clear that $\eta(t)$ is correlated at zero and $\pm\tau$, which manifests the non-tri*via*l properties of the recycling noise.

The fractional derivative has the contributions of damping force and restoring force [30], hence, we introduce the equivalent system:

$$
\ddot{x}(t)-u[-\varepsilon+\alpha_1\dot{x}^2(t)-\alpha_2\dot{x}^4(t)]\big[C(p)\dot{x}(t)+K(p)x(t)\big]+w^2x(t)=[\alpha+(1-\alpha)x(t)]\eta(t) \tag{6}
$$

where $C(p)$ and $K(p)$ are the coefficients of the equivalent damping and equivalent restoring forces of the fractional derivative ${}_0^C\mathrm{D}^p[x(t)]$, respectively.

Applying the equivalent method mentioned in [18], we get the ultimate forms of $C(p)$ and $K(p)$:

$$
K(p)=w^p\cos\frac{p\pi}{2},\quad C(p)=-w^{p-1}\sin\frac{p\pi}{2} \tag{7}
$$

Therefore, the equivalent Van der Pol oscillator associated with system (6) can be written:

$$
\begin{aligned}
\ddot{x}(t)+w_0{}^2x(t)-uw^p\cos\frac{p\pi}{2}[\alpha_1\dot{x}^2(t)-\alpha_2\dot{x}^4(t)]x(t)-w^{p-1}\sin\frac{p\pi}{2}\gamma\dot{x}(t)= \\
=[\alpha+(1-\alpha)x(t)]\eta(t)
\end{aligned}
\tag{8}
$$

where

$$
\gamma=u[-\varepsilon+\alpha_1\dot{x}^2(t)-\alpha_2\dot{x}^4(t)],\quad w_0{}^2=w^2+u\varepsilon w^p\cos\frac{p\pi}{2} \tag{9}
$$

Stationary PDF of the system amplitude

Linearizing the cubic and quintic stiffness terms and taking the undetermined damping and stiffness coefficients as functions of the system amplitude, the vibrational structure of the equivalent system can be rewritten [31]:

$$
\begin{aligned}
\ddot{x}(t)+w_0{}^2x(t)+C(a)\dot{x}(t)+K(a)x-uw^{p-1}\sin\frac{p\pi}{2}[-\varepsilon+\alpha_1\dot{x}^2(t)-\alpha_2\dot{x}^4(t)]\dot{x}(t)= \\
=[\alpha+(1-\alpha)x(t)]\eta(t)
\end{aligned}
\tag{10}
$$

To determine the coefficients $C(a)$ and $K(a)$ in eq. (10), the error between system (8) and system (10) is defined by:

$$
e=-uw^p\cos\frac{p\pi}{2}[\alpha_1\dot{x}^2(t)-\alpha_2\dot{x}^4(t)]x(t)-C(a)\dot{x}(t)-K(a)x(t) \tag{11}
$$

Assuming that the system (10) has the solution of the following form:

$$
x(t)=a(t)\cos\varphi(t),\quad \varphi(t)=w_0t+\theta(t) \tag{12}
$$

where ${w_0}^2 = w^2 + u\varepsilon w^p \cos(p\pi/2)$, using the generalized harmonic balance technique and making the error (11) minimized in the mean square sense, the undetermined coefficients $C(a)$ and $K(a)$ can be obtained [31]:

$$C(a) = -\frac{1}{aw_0\pi}\int_0^{2\pi}[-uw^p\cos\frac{p\pi}{2}[\alpha_1\dot{x}^2(t) - \alpha_2\dot{x}^4(t)]x(t) - K(a)x(t)]\sin\varphi\mathrm{d}\varphi = 0$$

$$K(a) = \frac{1}{a\pi}\int_0^{2\pi}\left\{-uw^p\cos\frac{p\pi}{2}[\alpha_1\dot{x}^2(t) - \alpha_2\dot{x}^4(t)]x(t) - C(a)\dot{x}(t)\right\}\cos\varphi\mathrm{d}\varphi =$$

$$= \frac{1}{8}\mu w^{p+2}\cos\frac{p\pi}{2}a^2(-2\alpha_1 + \alpha_2 w^2 a^2) \tag{13}$$

Substituting eq. (13) into eq. (10) gives the equivalent system:

$$\ddot{x}(t) + \Omega^2 x(t) - w^{p-1}\sin\frac{p\pi}{2}u[-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]\dot{x}(t) = [\alpha + (1-\alpha)x(t)]\eta(t) \tag{14}$$

where

$$\Omega^2 = w^2 + u\varepsilon w^p\cos\frac{p\pi}{2} + \frac{1}{8}\mu w^{p+2}\cos\frac{p\pi}{2}a^2(-2\alpha_1 + \alpha_2 w^2 a^2)$$

Assuming that system (14) has the solution of the periodic form, we introduce the following transformation [32]:

$$X = x(t) = a(t)\cos\Phi(t), \quad Y = \dot{x}(t) = -a(t)\Omega\sin\Phi(t), \quad \Phi(t) = \Omega t + \theta(t) \tag{15}$$

where Ω is natural frequency of the previous equivalent system (14), $a(t)$ and $\theta(t)$ represent the amplitude and phase processes of the system response, respectively, and they are both random processes.

Substituting eq. (15) into eq. (14), we obtain:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = F_{11}(a,\theta) + \mathrm{G}_{11}(a,\theta)\eta(t), \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = F_{21}(a,\theta) + G_{21}(a,\theta)\eta(t) \tag{16}$$

in which

$$F_{11}(a,\theta) = uw^{p-1}\sin\frac{p\pi}{2}a\sin^2\Phi(-\varepsilon + \alpha_1 a^2\Omega^2\sin^2\Phi - \alpha_2 a^4\Omega^4\sin^4\Phi)$$

$$F_{21}(a,\theta) = uw^{p-1}\sin\frac{p\pi}{2}\sin\Phi\cos\Phi(-\varepsilon + \alpha_1 a^2\Omega^2\sin^2\Phi - \alpha_2 a^4\Omega^4\sin^4\Phi) \tag{17}$$

$$G_{11} = -\alpha\frac{\sin\Phi}{\Omega} - (1-\alpha)\frac{a\cos\Phi\sin\Phi}{\Omega}, \quad G_{21} = -\alpha\frac{\cos\Phi}{a\Omega} - (1-\alpha)\frac{\cos^2\Phi}{\Omega}$$

Equation (1) can be treated as the Stratonovich stochastic differential equation, and by adding the relevant Wong-Zakai correction term, we transform it into the corresponding Ito stochastic differential equation:

$$\begin{aligned}\mathrm{d}a &= \left[F_{11}(a,\theta) + F_{12}(a,\theta)\right]\mathrm{d}t + \mathrm{G}_{11}(a,\theta)\mathrm{d}B(t)\\ \mathrm{d}\theta &= \left[F_{21}(a,\theta) + F_{22}(a,\theta)\right]\mathrm{d}t + G_{21}(a,\theta)\mathrm{d}B(t)\end{aligned} \tag{18}$$

where $B(t)$ is the normalized Wiener process and:

$$F_{12}(a,\theta)=\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial G_{11}(a,\theta)}{\partial a}G_{11}(a,\theta,t+h)+\frac{\partial G_{11}(a,\theta)}{\partial \theta}G_{21}(a,\theta,t+h)\right]$$
$$R(h)\mathrm{d}h=\frac{\alpha\cos^2(\Phi)DS(w)}{a\Omega^2}+(1-\alpha)\frac{a(\sin^2 2\Phi+4\cos 2\Phi\cos^2\Phi)DS(w)}{4\Omega^2}$$
$$F_{22}(a,\theta)=\frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial G_{21}(a,\theta)}{\partial a}G_{11}(a,\theta,t+h)+\frac{\partial G_{21}(a,\theta)}{\partial \theta}G_{21}(a,\theta,t+h)\right]$$
$$R(h)\mathrm{d}h=-\frac{\alpha\sin(2\Phi)DS(w)}{a^2\Omega^2}-(1-\alpha)\frac{\sin(2\Phi)\cos^2\Phi DS(w)}{\Omega^2} \tag{19}$$

By stochastic averaging [33] of eq. (18) over Φ , we obtain the following averaged Ito equation:

$$\mathrm{d}a=m_1(a)\mathrm{d}t+\sigma_{11}(a)\mathrm{d}B(t),\quad \mathrm{d}\theta=m_2(a)\mathrm{d}t+\sigma_{21}(a)\mathrm{d}B(t) \tag{20}$$

where

$$m_1(a)=uw^{p-1}\sin\frac{p\pi}{2}\left(-\frac{\varepsilon}{2}a+\frac{3}{8}\Omega^2\alpha_1 a^3-\frac{5}{16}\Omega^4\alpha_2 a^5\right)+\frac{\alpha DS(w)}{2a\Omega^2}+(1-\alpha)\frac{3aDS(w)}{8\Omega^2}$$
$$\sigma_{11}{}^2(a)=\frac{\alpha^2 DS(w)}{\Omega^2}+(1-\alpha)^2\frac{a^2 DS(w)}{4\Omega^2},\ m_2(a)=0,\ \sigma_{21}{}^2(a)=\frac{\alpha^2 DS(w)}{a^2\Omega^2}+(1-\alpha)^2\frac{3DS(w)}{4\Omega^2} \tag{21}$$

Equations (20) and (21) show that da does not depend on θ, the averaged Ito equation of $a(t)$ is independent of $\theta(t)$ and that the random process $a(t)$ is a 1-D diffusion process.

Thus, the reduced Fokker-Planck-Kolmogorov equation of $a(t)$ can be written:

$$0=-\frac{\partial}{\partial a}\left[m_1(a)\rho(a)\right]+\frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma_{11}{}^2(a)\rho(a)] \tag{22}$$

The boundary conditions are:

$$\rho(a)=c,\quad c\in(-\infty,+\infty),\quad \text{as } a=0$$
$$\rho(a)\to 0,\quad \frac{\partial\overline{p}}{\partial a}\to 0,\quad \text{as } a\to\infty \tag{23}$$

Based on the boundary conditions given in eq. (23), the amplitude stationary PDF can be obtained:

$$\rho(a)=\frac{C}{\sigma_{11}{}^2(a)}\exp\int_0^a\frac{2m_1(u)}{\sigma_{11}{}^2(u)}\,\mathrm{d}u \tag{24}$$

where C is the normalized constant that satisfies:

$$C=\left(\int_0^{\infty}\left\{\frac{1}{\sigma_{11}{}^2(a)}\exp\left[\int_0^a\frac{2m_1(u)}{\sigma_{11}{}^2(u)}\,\mathrm{d}u\right]\right\}\mathrm{d}a\right)^{-1} \tag{25}$$

Substituting eq. (21) into eq. (24), we get the explicit expression of stationary PDF of the system amplitude a:

$$\rho(a)=\frac{4C\Delta_1}{D\Delta_2}a^{\frac{4\alpha}{\Delta_2}}\exp\left[-\frac{\Delta_1\Delta_3+18D(\alpha-1)a^2}{12D\Delta_2}\right] \tag{26}$$

where C is the normalization constant and:

$$\Delta_1=\frac{\Omega^2}{S(w)},\quad \Delta_2=(a^2+4)\alpha^2-2a^2\alpha+a^2$$
$$\Delta_3=uw^{p-1}\sin\frac{p\pi}{2}(24\varepsilon a^2-9\alpha_1\Omega^2a^4+5\alpha_2\Omega^4a^6) \tag{27}$$
$$\Omega^2=w^2+u\varepsilon w^p\cos\frac{p\pi}{2}+\frac{1}{8}\mu w^{p+2}\cos\frac{p\pi}{2}a^2(-2\alpha_1+\alpha_2w^2a^2)$$

Stochastic P-bifurcation of the system amplitude

Stochastic P-bifurcation means that the changes in number of the stationary PDF curve peaks. To obtain the critical parametric conditions for stochastic P-bifurcation, we analyze the influences of parameters on the system stochastic P-bifurcation by using the singularity theory in this section.

For the sake of convenience, $\rho(a)$ is expressed by:

$$\rho(a)=\frac{4C}{D}R(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)\exp[Q(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)] \tag{28}$$

in which:

$$R(a,D,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3)=\frac{\Omega^2}{[(a^2+4)\alpha^2-2a^2\alpha+a^2]S(w)}a^{\frac{4\alpha}{(a^2+4)\alpha^2-2a^2\alpha+a^2}}$$
$$Q(a,D,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3)=$$
$$=\frac{\Omega^2uw^{p-1}\sin\frac{p\pi}{2}(24\varepsilon a^2-9\alpha_1\Omega^2a^4+5\alpha_2\Omega^4a^6)+18D(\alpha-1)a^2S(w)}{12D\Delta_2S(w)} \tag{29}$$
$$\Omega^2=w^2+u\varepsilon w^p\cos\frac{p\pi}{2}+\frac{1}{8}\mu w^{p+2}\cos\frac{p\pi}{2}a^2(-2\alpha_1+\alpha_2w^2a^2)$$

Based on the singularity theory [34], the stationary PDF of the system amplitude needs to satisfy:

$$\frac{\partial\rho(a)}{\partial a}=0,\quad \frac{\partial^2\rho(a)}{\partial a^2}=0 \tag{30}$$

Substituting eq. (28) into eq. (30), we obtain:

$$H=R'+RQ'=0, R''+2R'Q'+RQ''+RQ'^2=0 \tag{31}$$

where H is the condition for the changes in number of the PDF curve peaks.

In this part, the influences of p, τ, and D on the system are investigated, since the relationship of the 3-D surface is not easy to describe and display, here we only give the 2-D section of the transition set to represent the influences on the order of the fractional derivative and the noise intensity D, and then on the correlated time, τ, in the recycling noise and the noise intensity, D, respectively, below, without loss of generality, the parameters are taken as $\alpha_1 = 2.45$, $\alpha_2 = 4.6$, $w = 1$, $\varepsilon = 0.1$, $u = 1$, $k = 1$, and $\alpha = 1$, which means that the system is driven by purely additive Gaussian colored noise.

Taking (*p, D*) as unfolding parameters

Without losing generality, we take the correlated time, τ, in the recycling noise as $\tau = 0.5$, and according to eqs. (29) and (31), we obtain the transition set for the system stochastic P-bifurcation with the unfolding parameters p and D shown in fig. 1.

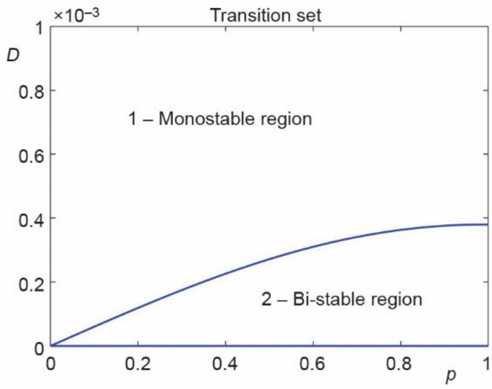

Figure 1. Transition set curves (taking *p* and *D* as the unfolding parameters)

Based on the singularity theory, the topological structures of the stationary PDF curves of different points (p, D) in the same area are qualitatively identical. By taking a point (p, D) in each area, we can obtain all varieties of the system stationary PDF curves that are qualitatively different. The unfolding parametric plane about p and D is divided into three sub-areas by the transition set curve. For the sake of convenience, each area in fig. 1 is marked with a number.

We first analyze the stationary PDF of amplitude $\rho(a)$ for a point (p, D) in each of the three sub-areas of fig. 1, and then compare the analytical solutions with the numerical data obtained by Monte-Carlo simulation from the original system (3) using the numerical method for fractional derivative [20]. The corresponding results are shown in fig. 2.

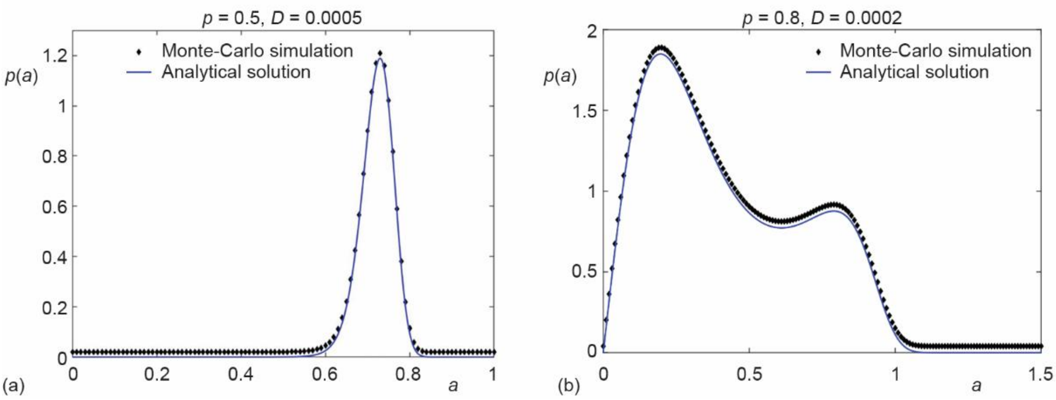

Figure 2. The PDF of *p*(*a*) (taking *p* and *D* as the unfolding parameters) in region 1 and region 2 of fig. 1; (a) parameter (*p*, *D*) in region 1 and (b) parameter (*p*, *D*) in region 2

From fig. 1 we can see that, the parameter area where the PDF $\rho(a)$ occurs bi-modal is surrounded by two curves. When the parameter (p, D) is taken as $p = 0.5$, $D = 0.0005$ in area 1, fig. 2(a), the PDF $\rho(a)$ has a stable limit cycle. When the parameter (p, D) is taken as $p = 0.8$, $D = 0.0002$ in area 2, fig. 2(b), the PDF $\rho(a)$ has a stable limit cycle far away from the origin and the probability is not zero near the origin, there are both the limit cycle and equilibrium in the system simultaneously.

Taking (τ, D) as unfolding parameters

Without losing generality, we take the order p of the fractional derivative as $p = 0.5$, and based on eqs. (29) and (31), we obtain the transition set for the system stochastic P-bifurcation with the unfolding parameters τ and D shown in fig. 3.

As shown in fig. 3, the unfolding parametric plane is segmented into three geometrically separated zones through the transition set curve. Based on the singularity analysis theory discussed previously, the PDF curves $\rho(a)$ for different parameters (τ, D) in the identical region exhibit qualitatively similar topological structures.

We discuss the PDF $\rho(a)$ for a given point (τ, D) in each sub-region in fig. 3. Subsequently, we contrast the theoretical outcomes with the Monte-Carlo simulation of the initial system (3) utilizing the numerical simulation technique of fractional derivative [20]. The corresponding outcomes are shown in fig. 4.

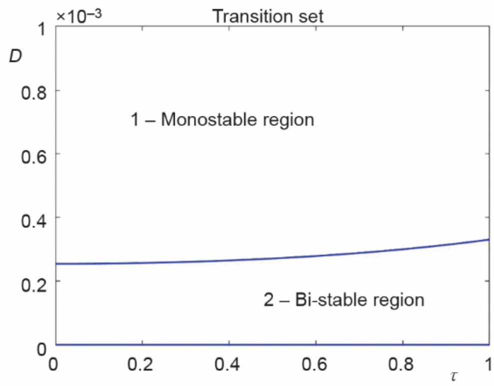

Figure 3. Transition set curves (taking τ and D as the unfolding parameters)

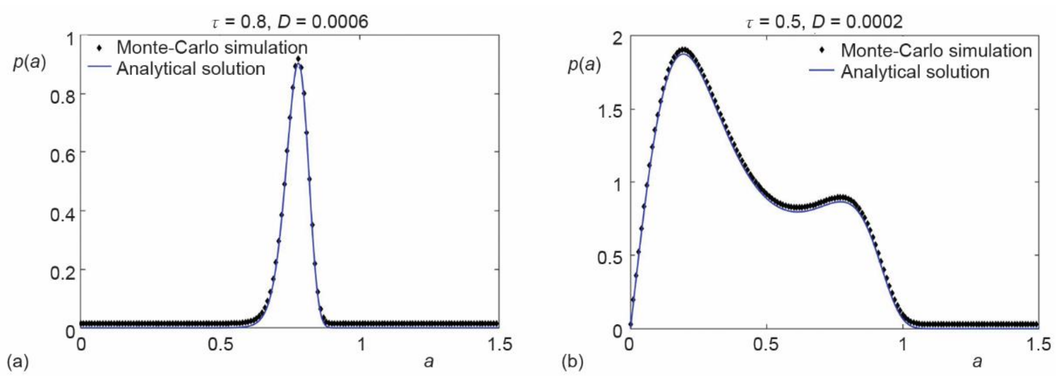

Figure 4. The PDF of $p(a)$ (taking τ and D as the unfolding parameters) in region 1 and region 2 of fig. 3; (a) parameter (τ, D) in region 1 and (b) parameter (τ, D) in region 2

In fig. 4, when the folding parameters (τ, D) are adopted in region 1, the PDF curve $\rho(a)$ of the system amplitude exhibits a prominent peak located at a significant distance from the origin as shown in fig. 4(a). Thus, the system only has a limit cycle (large amplitude motion) at this time. In region 2, the PDF curve $\rho(a)$ also has a remarkable peak near the original point, and the probability far away from the origin point is non-zero as displayed in fig. 4(b). A limit cycle and an equilibrium point exist in the given system (3) simultaneously at this moment.

Apparently, the stationary PDF $\rho(a)$ in any two adjacent areas in figs. 1 and 3 are very qualitatively different. Regardless of the exact values of the unfolding parameters, if they cross any line in this figure, the system will demonstrate stochastic P-bifurcation behavior. Therefore, the transition set curves are just the critical parametric conditions of the system stochastic P-bifurcation. The analytic results shown in fig. 2 are well consistent with those numerical results obtained by Monte-Carlo simulation from the original system (3), further verifying the theoretical analysis and showing that it is feasible to use the methods in this paper to analyze the stochastic P-bifurcation behavior of fractional order systems.

Compared with the integral-order controllers [35], the fractional-order controllers have the better dynamic performances and robustness [36]. In the past several years, various fractional-order controllers have been developed [37-40]. In the previous analysis we obtained the areas where the stochastic P-bifurcation occurs in system (3), which can make the system switch between mono-stable and bi-stable states by selecting the corresponding unfolding parameters. This could provide theoretical guidance for the analysis and design of the fractional order controllers.

Conclusions

In this paper, we investigated the bi-stable stochastic P-bifurcation of a generalized fractional Van der Pol system under the influence of additive Gaussian colored noise. A comprehensive investigation was conducted into the impacts of various parameters, including the order of the fractional derivative, the correlated time, and the intensity of recycling noise on the system. The utilization of the minimal mean square error principle resulted in the effective conversion of the original system into an equivalent integer-order system. Subsequently, by employing the stochastic averaging method, we derived the stationary PDF of the system amplitude. In accordance with the precepts of singularity theory, the critical parametric conditions for the stochastic P-bifurcation of the system were ascertained. According to these critical parametric conditions, the system response to different states can be manipulated by choosing appropriate unfolding parameters, which offers theoretical guidance for the design of such systems. The observed congruence between the Monte-Carlo simulation results and the analytical solutions serves to substantiate the veracity of our theoretical analysis. The results of the study indicate that both the order of the fractional derivative and the noise intensity can trigger stochastic P-bifurcation in the system. Furthermore, by selecting appropriate unfolding parameters, the number of peaks in the stationary PDF curves of the system amplitude can be adjusted from two to one. This also demonstrates the feasibility of the method outlined in this paper for analyzing the stochastic P-bifurcation behaviors of non-linear oscillators with fractional derivatives.

In the future, the research community may wish to extend the present study to include more complex, multidimensional fractional systems. In addition, the consideration of non-Gaussian noise sources and time-varying parameters may prove beneficial. As proposed by Chen *et al.* [41], a multi-role collaborative framework for structural damage identification considering measurement noise effects was developed. Integrating multiple complementary techniques could enhance the accuracy and robustness of the analysis for fractional systems. For instance, the integration of sophisticated machine learning algorithms or neural networks [42, 43] with fractal theory [44, 45] has the potential to unveil hitherto unexplored insights into the intricate stochastic dynamics of such systems. It is hypothesized that this will result in more precise predictions of system behaviors under a wider range of real-world conditions. This will further enrich the theoretical and practical applications of fractional-order systems in various fields, including physics, engineering, and biology.

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