EXACT TRAVELING WAVE SOLUTION OF DULLIN-GOTTWALD-HOLM EQUATION WITH CONFORMABLE FRACTIONAL DERIVATIVE

by

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The present study focuses on the Dullin-Gottwald-Holm equation, which incorporates a conformable fractional derivative. Employing the two-scale transform and traveling wave transformation, along with the integration of hyperbolic functions and Jacobi elliptic functions, a comprehensive and systematic investigation is conducted. Consequently, the precise traveling wave solutions of this equation have been derived. A thorough discussion is presented on the impacts of various parameters on the properties of the solutions. Furthermore, the approach employed in this paper demonstrates considerable promise in addressing a broader range of non-linear fractional differential equations. It is anticipated that this will pioneer novel concepts and methods in related research areas.

Keywords: *Dullin-Gottwald-Holm equation, conformable fractional derivative, exact travelling wave solutions*

Introduction

The Dullin-Gottwald-Holm (DGH) equation plays a crucial role in fluid dynamics [1, 2]. Its classical form is expressed:

$$\frac{\partial}{\partial t}\left(u-\alpha_2^2\frac{\partial^2 u}{\partial x^2}\right)+\alpha_1\frac{\partial u}{\partial x}-\alpha_2^2\left(u\frac{\partial^3 u}{\partial x^3}+2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right)+3u\frac{\partial u}{\partial x}+\alpha_3\frac{\partial^3 u}{\partial x^3}=0 \tag{1}$$

where u is the fluid velocity field, α_i (i = 1, 2, 3) are some parameters, and t and x denote time variable and space variable, respectively.

The classical DGH equation came into being as researchers dedicated themselves to understanding and describing the intricate wave phenomena within fluid systems. Since then, it has firmly established itself as a central focus in the realm of non-linear evolution equation research. A multitude of studies delving into its properties and solutions have been meticulously documented in the literature, as comprehensively detailed in [3-5]. These investigations have not only enhanced our understanding of the DGH equation itself but have also contributed significantly to the broader field of non-linear dynamics, offering valuable insights into the behavior of complex fluid-wave interactions.

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In recent years, the field of fractional derivatives has witnessed remarkable development [6, 7]. Fractional-order differential equations, which incorporate the non-local and memory-effect characteristics of fractional derivatives, have proven to be powerful tools for modeling real-world problems [8]. These equations are especially useful in scenarios where the history of a process significantly influences its current state, and traditional local-state-based models fall short. As a result, many physical phenomena can be described more accurately when fractional-order derivatives are incorporated into non-linear differential equations. Given the wide range of applications of the DGH equation in real-world scenarios, such as in the study of water waves and fluid-solid interactions, an increasing number of scholars have turned their attention to fractional DGH equations [9, 10], and the fractional soliton theory have become the hottest topic in mathematics and thermal science [11, 12].

In this paper, we focus on a specific DGH equation involving the conformable fractional derivative [13], as shown:

$$\mathrm{D}_t^{\lambda}\left(u-\alpha_2^2\mathrm{D}_x^{2\beta}u\right)+\alpha_1\mathrm{D}_x^{\beta}u-\alpha_2^2\left(u\mathrm{D}_x^{3\beta}u+2\mathrm{D}_x^{\beta}u\mathrm{D}_x^{2\beta}u\right)+3u\mathrm{D}_x^{\beta}u+\alpha_3\mathrm{D}_x^{3\beta}u=0 \tag{2}$$

where $0<\lambda\le 1$, $0<\beta\le 1$, D_t^{λ} is the time conformable fractional derivative of order λ, λ – the space conformable fractional derivative of order λ, and λ are some parameters. The fractional order λ is relative to two-scale fractal dimensions [14-17], and λ is relative to fractal time. It was elucidated that when space is fractional, time must be fractional, this is called as fractional spatio-temporal relation [18].

Our primary objective is to obtain the exact travelling wave solutions of eq. (2). These solutions can provide deeper insights into the physical phenomena described by fractional DGH equations, such as the behavior of waves in complex fluid media.

Preliminaries

Conformable fractional derivative

In this section, we briefly introduce the key concepts of conformable fractional calculus theory, with more in-depth details in [10] and references cited thereby.

Definition 1. Let $\omega\in(0, 1)$ and f:[0, ∞) → R. The conformable fractional derivative of f of order α is defined by:

$$\mathrm{D}_t^{\omega}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon\,t^{1-\omega})-f(t)}{\varepsilon} \tag{3}$$

for all $t>0$. We often denote the conformable fractional derivative of f of order ω as $f^{(\omega}$ instead of $\mathrm{D}_t^{\omega}f(t)$. If the conformable fractional derivative of f of order ω exists, we say that f is ω-differentiable. If f is ω-differentiable on some interval (0, a) with $a>0$ and $\lim_{t\to 0^+}f^{(\omega)}(t)$ exists, then we define:

$$f^{(\omega)}(0)=\lim_{t\to 0^+}f^{(\omega)}(t) \tag{4}$$

Theorem 1. If a function $f(x)$:[0, ∞) → R is ω-differentiable at $t_0>0$, then f is continuous at t_0.

Theorem 2. Let $\omega\in(0, 1]$ and assume f and g to be ω-differentiable. Then:

$$\mathrm{D}_t^{\omega}(af+bg)=a\mathrm{D}_t^{\omega}f+b\mathrm{D}_t^{\omega}g \quad \text{for all} \quad a,b\in R \tag{5}$$

$$\mathrm{D}_t^{\omega}(fg) = g\mathrm{D}_t^{\omega} f + f\mathrm{D}_t^{\omega} g \tag{6}$$

$$\mathrm{D}_t^{\omega}\frac{f}{g} = \frac{g\mathrm{D}_t^{\omega} f - f\mathrm{D}_t^{\omega} g}{g^2} \tag{7}$$

If $f(x)$ is differentiable at a point $t > 0$, then we have:

$$\mathrm{D}_t^{\omega} f(t) = \frac{t^{1-\omega}\mathrm{d}f}{\mathrm{d}t} \tag{8}$$

Definition 2. Let $\omega \in (0, 1)$ and f:[0, ∞) → R. The conformable fractional integral of f of order ω from a to t, denoted by ${}_aI_t^{\omega}(f)$, is defined by:

$${}_aI_t^{\omega}(f) = \int_a^t \tau^{\omega-1} f(\tau)\,\mathrm{d}\tau = \int_a^t f(\tau)\,\mathrm{d}_{\omega}\tau \tag{9}$$

where the previosu integral is usual improper Riemann integral.

Jacobi elliptic functions

In this section, we revisit fundamental definitions and properties of Jacobi elliptic functions [19, 20]. Let k be a number in (0, 1]. Consider the following integral:

$$\xi = \xi(z,k) = \int_0^z \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{10}$$

The Jacobi elliptic sine function $z = sn(\xi,k)$ is defined as the inverse function of $\xi = \xi(z,k)$. Moreover, the Jacobi elliptic cosine function $z = cn(\xi,k)$ and the Jacobi elliptic delta function $z = \mathrm{d}n(\xi,k)$ are defined respectively as $cn(\xi,k) = [1 - sn^2(\xi,k)]^{1/2}$, and $\mathrm{d}n(\xi,k) = [1 - k^2 sn^2(\xi,k)]^{1/2}$.

Proposition 1. For the derivatives of the Jacobi elliptic functions, we have:

$$\frac{\mathrm{d}}{\mathrm{d}\xi} sn(\xi,k) = cn(\xi,k)\,\mathrm{d}n(\xi,k) \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} cn(\xi,k) = -sn(\xi,k)\,\mathrm{d}n(\xi,k) \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \mathrm{d}n(\xi,k) = -k^2 cn(\xi,k) sn(\xi,k) \tag{13}$$

Proposition 2. As k approaches 0 from the right, we have:

$$sn(\xi,k) \to \sin(\xi), \quad cn(\xi,k) \to \cos(\xi), \quad \mathrm{d}n(\xi,k) \to 1 \tag{14}$$

and as k approaches 1 from the left, we have:

$$sn(\xi,k) \to \tanh(\xi), \quad cn(\xi,k) \to \mathrm{sech}(\xi), \quad \mathrm{d}n(\xi,k) \to \mathrm{sech}(\xi) \tag{15}$$

Some Lemmas

In this section, we give some *Lemmas* which will be used in the rest of the paper.

Lemma 1. Let:

$$F(\xi)=\frac{-c_1+\sqrt{4c_0c_2-c_1^2}\sinh\left(\sqrt{c_2}\xi\right)}{2c_2}$$

Then $F(\xi)$ satisfies the equation:

$$c_2F^2+c_1F+c_0=\left(\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)^2,\quad c_2>0$$

Proof. We have:

$$\frac{\mathrm{d}F}{\mathrm{d}\xi}=\frac{\sqrt{4c_0c_2-c_1}}{2c_2}\sqrt{c_2}\cosh\left(\xi\sqrt{c_2}\right)$$

So that:

$$\left(\frac{\mathrm{d}F}{\mathrm{d}\xi}\right)^2=\frac{4c_0c_2-c_1}{4c_2}\cosh^2\left(\xi\sqrt{c_2}\right)$$

On the other hand:

$$c_2F^2+c_1F+c_0=c_0+\frac{c_1^2}{4c_2}-\frac{c_1^2}{2c_2}+\frac{4c_0c_2-c_1^2}{2c_2}\sinh^2\left(\xi\sqrt{c_2}\right)=\frac{4c_0c_2-c_1^2}{4c_2}\left[1+\sinh^2\left(\xi\sqrt{c_2}\right)\right]$$

Thus we have completed the proof of *Lemma 1*.

Lemma 2. Let:

$$G(\xi)=r_2+(r_1-r_2)cn^2\left[\sqrt{\frac{A}{4}(r_1-r_3)}(\xi),\quad\sqrt{\frac{r_1-r_2}{r_1-r_3}}\right]$$

Then $G(\xi)$ satisfies the equation:

$$-A(G-r_1)(G-r_2)(G-r_3)=\left(\frac{\mathrm{d}G}{\mathrm{d}\xi}\right)^2,\quad A>0,\quad r_3\le r_2\le r_1$$

Proof. The proof is straightforward by eqs. (11)-(13).

The solutions of eq. (2)

In this section, we seek the exact travelling wave solutions of eq. (2).

Firstly, by using two-scale transform [21-23]:

$$X=\frac{x^\beta}{\beta},\quad T=\frac{t^\lambda}{\lambda}\tag{16}$$

Equation (2) can be converted into:

$$\frac{\partial u}{\partial T}+\alpha_1\frac{\partial u}{\partial X}+3u\frac{\partial u}{\partial X}-\alpha_2^2\left(\frac{\partial^3 u}{\partial X^2\partial T}+u\frac{\partial^3 u}{\partial X^3}+2\frac{\partial u}{\partial X}\frac{\partial^2 u}{\partial X^2}\right)+\alpha_3\frac{\partial^3 u}{\partial X^3}=0 \tag{17}$$

In order to seek exact travelling wave solutions of eq. (17), we introduce the traveling wave transformation*:*

$$u(X,T)=u(\xi),\quad \xi=lX+\omega T+\eta \tag{18}$$

where *l,* ω, and η are constants.

Then, the transformation is substituted into (17) to produce the following form:

$$(l\alpha_1+\omega)u_\xi-\alpha_2^2(l^2\omega u_{\xi\xi\xi}+l^3uu_{\xi\xi\xi}+2l^3u_\xi u_{\xi\xi})+3luu_\xi+\alpha_3l^3u_{\xi\xi\xi}=0 \tag{19}$$

Direct integrating of eq. (19) in ξ yields:

$$(\omega+\alpha_1l)u+(\alpha_3l^3-\alpha_2^2\omega l^2)u_{\xi\xi}-\alpha_2^2l^3uu_{\xi\xi}-\frac{\alpha_2^2}{2}l^3u_\xi^2+\frac{3}{2}lu^2=c_1 \tag{20}$$

Let du/dξ = v. From eq. (20), we obtain:

$$(\alpha_3l^3-\omega\alpha_2^2l^2-\alpha_2^2l^3u)v^2=c_2+c_1u-(\omega+l\alpha_1)u^2-lu^3 \tag{21}$$

where c_1 and c_2 are constants.

Next, we consider two cases:

Case 1. If $\alpha_2=0$, from (21) we have:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^2=\frac{-1}{\alpha_3l^2}f(u) \tag{22}$$

where $f(u)=u^3+(\alpha_1+\omega l^{-1})u^2-c_1u-c_2$.

Assume that u_3, u_2, and u_1 satisfy the equation $f(u)=0$. By *Lemma 2*, we can obtain:

$$u=u_2+(u_1-u_2)cn^2\left[\sqrt{\frac{u_1-u_3}{4\alpha_3l^2}}(lX+\omega T+\eta),\quad \sqrt{\frac{u_1-u_2}{u_1-u_3}}\right] \tag{23}$$

Taking $u_2=u_3=0$, we get the solution of eq. (17) :

$$u=(-\alpha_1-\omega l^{-1})\operatorname{sech}^2\left[\sqrt{\frac{-1}{4\alpha_3l^2}(\alpha_1+\omega l^{-1})}(lX+\omega T+\eta)\right] \tag{24}$$

From eq. (16), we obtain the exact solution of eq. (2) in the form:

$$u=(-\alpha_1-\omega l^{-1})\operatorname{sech}^2\left[\sqrt{\frac{-1}{4\alpha_3l^2}(\alpha_1+\omega l^{-1})}\left(\frac{x^\beta l}{\beta}+\frac{t^\lambda\omega}{\lambda}+\eta\right)\right] \tag{25}$$

Case 2. If $\alpha_2\neq 0$, from (21) we can get:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^2 = \frac{1}{\alpha_2^2 l^2}u^2 + \frac{\alpha_1\alpha_2^2+\alpha_3}{\alpha_2^4 l^2}u + \alpha_1\alpha_3 \tag{26}$$

By *Lemma 1*, we get:

$$u = -\frac{\alpha_1\alpha_2^2}{2\alpha_2^2} + \frac{|\alpha_1\alpha_2^2-\alpha_3|l}{2}\sinh\left[\frac{1}{\alpha_2 l}(lX+\omega T+\eta)\right] \tag{27}$$

Finally, from eq. (16), we obtain the exact solution of eq. (2) in the form:

$$u = -\frac{\alpha_1\alpha_2^2}{2\alpha_2^2} + \frac{|\alpha_1\alpha_2^2-\alpha_3|l}{2}\sinh\left[\frac{1}{\alpha_2 l}\left(\frac{lx^\beta}{\beta}+\frac{\omega t^\lambda}{\lambda}+\eta\right)\right] \tag{28}$$

Conclusions

In this study, we have successfully derived the exact traveling wave solutions for a class of DGH equations with conformable fractional derivatives by leveraging *Lemmas 1* and *2*. These solutions are considered novel, as they have not been previously documented in the extant literature. These findings offer novel insights into the behavior of waves described by such fractional-order equations, which is particularly significant for understanding complex physical phenomena in fluid dynamics and related fields.

Beyond the specific solutions obtained, the method employed in this paper represents a valuable approach. This finding suggests the possibility of extending these methods to other non-linear fractional differential equations. This finding indicates that the present research can function as a foundational basis for subsequent investigations within the broader domain of fractional-order differential equations. Subsequent studies may have the capacity to augment the present undertaking by exploring more intricate equations, examining the stability and qualitative properties of solutions, and investigating their relationships with real-world applications. This research contributes to the expanding corpus of knowledge in the domain of fractional-order equations and their applications.

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