VARIATIONAL APPROACH FOR THE FRACTAL GENERALIZED BOGOYAVLENSKY-KONOPELCHENKO EQUATION

by

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This paper explores a fractal generalized Bogoyavlensky-Konopelchenko equation (gBK) defined by He's fractal derivative. Employing the variational approach in conjunction with the two-scale fractal transformation has yielded the fractal wave solutions. Furthermore, we offer remarks on the variational formulation of the conventional gBK equation presented in previous literature. Notably, we have identified two novel solutions: the fractal periodic wave solution and the fractal bright soliton solution. These solutions have not been previously explored in the existing literature. The propagation behavior of these fractal wave solutions is vividly demonstrated through 3-D figures with diverse fractal dimensions and amplitudes. The findings of the present study contribute to the theoretical research on the gBK equation and provide valuable insights for future studies in the field of fractal differential equations.

Keywords: *Bogoyavlensky-Konopelchenko equation, variational approach, solution, two-scale fractal transformation*

Introduction

In recent decades, non-linear partial differential equations (NPDE) have garnered significant attention owing to their extensive applications across diverse engineering and scientific domains. These include fluid mechanics, plasma physics, mechanical vibration, and numerous other fields. For instance, Yu *et al.* [1] explored N-soliton solutions to the Bogoyavalenskii-Schiff equation and the soliton solution in (3+1)-D, contributing to the understanding of solitons in complex systems. Pervaiz and Aziz [2] used the Haar wavelet approximation to solve cubic non-linear Schrodinger equations, providing a numerical approach for such NPDE. Soliman [3] proposed the modified extended tanh-function method for Burgers-type equations, expanding the toolkit for solving specific NPDE. Morris and Leach [4] focused on symmetry reductions and solutions of the Zoomeron equation, highlighting the role of symmetry in NPDE analysis. Kumar *et al.* [5] studied elliptic and solitary wave solutions for multiple equation systems, enriching the knowledge of wave solutions in different contexts. Sendi *et al.* [6] applied the ITEM to solve non-linear evolution equations in fluid mechanics, demonstrating the practical use of methods in a specific field. Asad *et al.* [7] investigated the asymmetric variation of a finite-mass harmonic-like oscillator, which is relevant to the study of non-linear dynamics related to NPDE. He *et al.* [8] analyzed the pull-down instability of quadratic non-linear oscillators, providing insights into oscillator-related NPDE. Lu and Ma

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[9] explored the fractal modification of an attachment oscillator, which is related to the application of fractal concepts in NPDE. Lu [10] used the variational approach for the (3+1)-D shallow-water wave equation, offering a solution method for a particular NPDE. Wang [11] focused on intelligent nanomaterials for solar energy harvesting, showing an interdisciplinary application related to NPDE in a broader sense. He *et al.* [12] studied plants distribution in a fractal space, which is relevant to the application of fractal theory in NPDE-related research. Zheng *et al.* [13] conducted a numerical analysis of fractional non-linear vibration of a micro/nano beam-based micro-electromechanical system, which is related to the numerical study of NPDE in engineering structures.

A variety of efficient methods, grounded in analytical and numerical techniques, have been developed to obtain different types of solutions for NPDE. These solutions encompass solitary waves, soliton-type, lump-type, FCP solutions, interaction solutions, and more. Dong *et al.* [14] studied solitary waves, homoclinic breather waves, and rogue waves of the (3+1)-D Hirota bilinear equation, contributing to the understanding of complex wave solutions. Alazman [15] analyzed the non-linear complex dynamics and solitary waves of the (3+1)-D non-linear extended quantum Zakharov-Kuznetsov equation, providing insights into a specific NPDE. Rady and Khalfallah [16] focused on soliton solutions for Boussinesq-Burgers equations, adding to the research on soliton-type solutions. Zhou *et al.* [17] investigated auto-Backlund transformations and soliton solutions for a (3+1)-D Korteweg-de Vries-Calogero-Bogoyavlenskii-Schif equation, exploring transformation-based solution methods. Gao *et al.* [18] studied Backlund transformation, multiple-wave solutions, and lump solutions for a (3+1)-D non-linear evolution equation, expanding the scope of solution-seeking for NPDE. Ma [19] searched for lump solutions to a combined fourth-order non-linear PDE in (2+1)-D, focusing on a particular type of solution. Jia [20] explored soliton molecules and few-cycle-pulse solitons for the generalized Konopelchenko-Dubrovsky-Kaup-Kupershmidt equation, contributing to the understanding of complex soliton structures. Xu, *et al.* [21] studied rogue waves for the (2+1)-D Kadomtsev-Petviashvili equation, furthering the research on rogue-wave solutions. Ma and Zhu [22] solved the (3+1)-D generalized KP and BKP equations using the multiple exp-function algorithm, presenting an effective solution method.

In a study by Chen and Ma [23], a (2+1)-D generalized Bogoyavlensky- Konopelchenko (gBK) equation was investigated, which is presented:

$$w_t+\alpha(6ww_x+w_{xxx})+\beta[w_{xxy}+3(ww_y+w_xu_y)]+\gamma_1w_x+\gamma_2w_y+\gamma_3u_{yy}=0 \tag{1}$$

where $w = u_x$, and α, β, γ_i (i = 1, 2, 3) are given constants. By substituting $w = u_x$ into (1), we can reformulate (1) in the following equivalent form:

$$u_{xt}+\alpha(6u_xu_{xx}+u_{xxxx})+\beta[u_{xxxy}+3(u_xu_{xy}+u_{xx}u_y)]+\gamma_1u_{xx}+\gamma_2u_{xy}+\gamma_3u_{yy}=0 \tag{2}$$

The gBK eq. (2) is a generalization of the Bogoyavlensky-Konopelchenko equation, which can be used to model the interaction behavior between a long wave propagating along the x-direction and a Riemann wave along the y-direction [24, 25]. Lump solutions of the non-linear eq. (2) was given by using MAPLE symbolic computations together with its Hirota bilinear form [23, 26]. Similar technique was suggested by Chen and Ma [27] to find one-wave type and two-wave solutions and polynomial solutions. Li *et al.* [28] obtained lump-type solutions and lump solutions with the help of Hirota bilinear method and the ansatz technique. Multiple rogue wave solutions and solitary solutions were provided by using the multiple lump solutions method and He's semi-inverse variational principle, respectively [29]. Different types of wave or soliton solutions were also given by means of the Hirota bilinear method and the variational principle [30]. Lump-type solutions and localized interaction solutions were investigated by Zhang *et al.* [31] and Ma *et al.* [32]. There are other improvements about the BK-type equations, such as the fractional BK equation and the BK equation with variable coefficients, see [33-35]. Reviewing the recent work about the gBK equations [23, 27-32], the numerical or analytical approaches are based upon that different solutions to (2) propagate along the smooth boundary. The topic about the behavior analysis of wave solutions or other solutions with non-smooth boundary is interesting, and requires further investigation. When different types of wave solutions to (2) travel along the non-smooth boundaries, or are observed under a small scale, this non-linear system may be invalid due to the solution function of (2) is non-differentiable about the time variable or the spatial variables x and y. In order to solve this issue, the fractal modification of the gBK equation can be suggested, which is the main motivation of this paper. There are two common fractal derivatives including the fractal Hausdorff derivative and He's fractal derivative which have been widely applied for modeling different engineering problems [36-39]. In this work, we will focus on the analysis of the fractal generalized Bogoyavlensky-Konopelchenko equation with He's fractal derivative:

$$\frac{\partial^r}{\partial t^r}\left(\frac{\partial^p u}{\partial x^p}\right)+\alpha\left(6\frac{\partial^p u}{\partial x^p}\frac{\partial^{2p} u}{\partial x^{2p}}+\frac{\partial^{4p} u}{\partial x^{4p}}\right)+\beta\left[\frac{\partial^q}{\partial y^q}\left(\frac{\partial^{3p} u}{\partial x^{3p}}\right)+3\frac{\partial^p u}{\partial x^p}\frac{\partial^q}{\partial y^q}\left(\frac{\partial^p u}{\partial x^p}\right)+3\frac{\partial^{2p} u}{\partial x^{2p}}\frac{\partial^q u}{\partial y^q}\right]+$$

$$+\gamma_1\frac{\partial^{2p} u}{\partial x^{2p}}+\gamma_2\frac{\partial^q}{\partial y^q}\left(\frac{\partial^p u}{\partial x^p}\right)+\gamma_3\frac{\partial^{2q} u}{\partial y^{2q}}=0 \tag{3}$$

where $0 < r, p, q \leq 1$ are three fractal dimensions about the direction of time t and space x or y, respectively. The fractal derivatives proposed by He [37, 38] are suggested to define the fractal operators $(\partial^r u)/(\partial t^r)$, $(\partial^p u)/(\partial x^p)$, and $(\partial^q u)/(\partial y^q)$:

$$\frac{\partial^r u(x,y,t_0)}{\partial t^r}=\Gamma(1+r)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{u(x,y,t)-u(x,y,t_0)}{(t-t_0)^r} \tag{4}$$

$$\frac{\partial^p u(x_0,y,t)}{\partial x^p}=\Gamma(1+p)\lim_{\substack{x-x_0\to\Delta x\\ \Delta x\neq 0}}\frac{u(x,y,t)-u(x_0,y,t)}{(x-x_0)^p} \tag{5}$$

$$\frac{\partial^q u(x,y_0,t)}{\partial y^q}=\Gamma(1+q)\lim_{\substack{y-y_0\to\Delta y\\ \Delta y\neq 0}}\frac{u(x,y,t)-u(x,y_0,t)}{(y-y_0)^q} \tag{6}$$

with the small scales Δt, Δx, and Δy in time and spatial spaces.

Generally speaking, it is difficult to directly obtain the fractal solutions to the fractal PDE. The difficulty for finding the solutions to the fractal gBK equation comes from the fractal operators in (3). By using the two-scale fractal transformation, (3) can be transformed to the original gBK equation. Then the conventional gBK equation is equivalently rewritten as an ODE via the wave transformation. The semi-inverse method proposed by He [40] is an efficient approach to give variational principles of various differential equations. This variational approach suggested by He is used for finding the fractal wave solutions to (3). The variational formulation is given, which can be seen as the corrections to the existing results in [29, 30]. The fractal solutions including the fractal periodic wave solution and the fractal briton soliton solution are given by considering the stationary condition of the variational operator. For illustrating the propagation behavior of the fractal wave solutions, we also consider the numerical analysis of the obtained solutions with various fractal dimensions and amplitudes. Sensitive analysis of the frequency, ω, is also investigated about different amplitudes, A. Finally, some conclusions and future work are given.

Variational approach for the fractal gBK equation

Consider the following two-scale fractal transformation proposed by He [37, 38]:

$$T = t^r, \quad X = x^p, \quad Y = y^q$$

The fractal eq. (3) can be rewritten as the following conventional gBK equation:

$$u_{XT} + \alpha(6u_X u_{XX} + u_{XXXX}) + \beta[u_{XXXY} + 3(u_X u_{XY} + u_{XX} u_Y)] + \gamma_1 u_{XX} + \gamma_2 u_{XY} + \gamma_3 u_{YY} = 0 \quad (8)$$

Then by using the wave transformation $\xi = k\left(X + Y - cT\right)$, and substituting it into (8), we have the following ordinary differential equation:

$$k^2(\alpha+\beta)u'''' + 6k(\alpha+\beta)u'u'' + (\gamma_1+\gamma_2+\gamma_3-c)u'' = 0 \quad (9)$$

We remark that the variational approach was considered to give the wave solutions to (2), and the variational formulation was given by [29, 30]:

$$\hat{J} = \int\left[2k(\alpha+\beta)(u')^3 - \frac{1}{2}k^2(\alpha+\beta)(u')^2 + \frac{1}{2}(\gamma_1+\gamma_2+\gamma_3-c)(u'')^2 + k^2(\alpha+\beta)u'u'''\right]\mathrm{d}\xi \quad (10)$$

Since the variational method is based upon the variational derivative [40]:

$$\frac{\delta J}{\delta u} = \frac{\partial J}{\partial u} - \frac{\partial}{\partial \xi}\frac{\partial J}{\partial u'} + \frac{\partial}{\partial \xi^2}\frac{\partial J}{\partial u''} - \frac{\partial}{\partial \xi^3}\frac{\partial J}{\partial u'''} + \cdots \quad (11)$$

the stationary condition of (10) does not hold for the non-linear equation (8). This motivates us to provide the corrected results for the variational approach. By applying variational approach suggested by the semi-inverse method [41-47], we obtain the following variational formulation:

$$J\left(u\right) = \int\left[-\frac{1}{2}\left(\gamma_1+\gamma_2+\gamma_3-c\right)\left(u'\right)^2 - k\left(\alpha+\beta\right)\left(u'\right)^3 + \frac{1}{2}k^2\left(\alpha+\beta\right)\left(u''\right)^2\right]\mathrm{d}\xi \quad (12)$$

The stationary condition of (12) defined by (11) satisfies (2). Thus, (12) can be seen as the corrections to (10) [29, 30].

We first assume that the periodic wave solution to (8) can be formulated by:

$$u = A\cos\left(\omega\xi\right) \quad (13)$$

where A is a given amplitude, and ω is a unknown frequency determined later.

By substituting (13) into (12), we obtain the following function:

$$J\left(A\right)=\int_0^{T/4}\Bigg[-\frac{1}{2}\left(\gamma_1+\gamma_2+\gamma_3-c\right)A^2\omega^2\sin^2\left(\omega\xi\right)+k\left(\alpha+\beta\right)A^3\omega^3\sin^3\left(\omega\xi\right)+$$

$$+\frac{1}{2}k^2\left(\alpha+\beta\right)A^2\omega^2\cos^2\left(\omega\xi\right)\Bigg]\mathrm{d}\xi \tag{14}$$

The stationary condition for (14) given by [d*J*(*A*)]/(d*A*) = 0 leads to the following equation:

$$\int_0^{T/4}\left(c-\gamma_1-\gamma_2-\gamma_3\right)A\omega^2\sin^2\left(\omega\xi\right)+3k\left(\alpha+\beta\right)A^2\omega^3\sin^3\left(\omega\xi\right)+$$

$$+k^2\left(\alpha+\beta\right)A\omega^2\cos^2\left(\omega\xi\right)\Big]\mathrm{d}\xi=0 \tag{15}$$

Then the unknown frequency ω can be followed from (15) with $T = 2\pi$:

$$\omega=\frac{\displaystyle\int_0^{T/4}\Big[\left(\gamma_1+\gamma_2+\gamma_3-c\right)\sin^2\left(\omega\xi\right)-k^2\left(\alpha+\beta\right)\cos^2\left(\omega\xi\right)\Big]\mathrm{d}\xi}{\displaystyle\int_0^{T/4}\Big[3kA\left(\alpha+\beta\right)\sin^3\left(\omega\xi\right)\Big]\mathrm{d}\xi}=$$

$$=\frac{\pi}{8A}+\frac{\pi\left(\gamma_1+\gamma_2+\gamma_3-c\right)}{8Ak\left(\alpha+\beta\right)} \tag{16}$$

Therefore, we obtain the periodic wave solution of the fractal form:

$$u=A\cos\left[\left(\frac{k\pi}{8A}+\frac{\pi\left(\gamma_1+\gamma_2+\gamma_3-c\right)}{8A\left(\alpha+\beta\right)}\right)\left(x^p+y^q-ct^r\right)\right] \tag{17}$$

We then consider the briton soliton solution to (8) defined by:

$$u=A\operatorname{sech}\left(B\xi\right) \tag{18}$$

where *A* and *B* are two unknown constants.

By (12) and 18), it follows that:

$$J\left(A,B\right)=\frac{1}{30}A^2B(5c+Bk\left(\alpha+\beta\right)\Big[4A+7Bk-5\left(\gamma_1+\gamma_2+\gamma_3\right)\Big] \tag{19}$$

The stationary conditions are given by [∂*J*(*A*, *B*)]/(∂*A*) = 0 and [∂*J*(*A*, *B*)]/(∂*B*) = 0, which can be written as the following equations:

$$\frac{\partial J\left(A,B\right)}{\partial A}=\frac{1}{15}AB\Big[5\left(c-\gamma_1-\gamma_2-\gamma_3\right)+Bk\left(\alpha+\beta\right)\left(6A+7Bk\right)\Big]=0 \tag{20}$$

$$\frac{\partial J\left(A,B\right)}{\partial B}=\frac{1}{30}A^2\Big[5\left(c-\gamma_1-\gamma_2-\gamma_3\right)+Bk\left(\alpha+\beta\right)\left(8A+21Bk\right)\Big]=0 \tag{21}$$

When $(c-\gamma_1-\gamma_2-\gamma_3)(\alpha+\beta)>0$, the unknown parameters A and B can be followed from the previous systems:

$$A=\pm\sqrt{\frac{7(c-\gamma_1-\gamma_2-\gamma_3)}{\alpha+\beta}} \tag{22}$$

$$B=\mp\sqrt{\frac{c-\gamma_1-\gamma_2-\gamma_3}{7k^2(\alpha+\beta)}} \tag{23}$$

Thus, we have the following fractal briton soliton solution to (3):

$$u=\pm\sqrt{\frac{7(c-\gamma_1-\gamma_2-\gamma_3)}{\alpha+\beta}}\operatorname{sech}\left[\mp\sqrt{\frac{c-\gamma_1-\gamma_2-\gamma_3}{7k^2(\alpha+\beta)}}\left(x^p+y^q-ct^r\right)\right] \tag{24}$$

We remark that this fractal wave solutions including (17) and (24) are not touched in the existing works. The fractal space related with (3), (17) or (24) is always anisotropy, which implies that the fractal dimensions of *x*-, *y*-, and *t*-directions are different. The fractal form of the kinky-bright soliton solution and other soliton-type solution to the fractal eq. (3) can also be given by similar variational method. When $p=q=r=1$, the soliton-type solutions to (2) can be followed, which can be viewed as the corrected results in [29, 30].

Numerical results and discussions

We focus on considering the propagation behavior of the fractal periodic wave solutions to (3), where various fractal dimensions and amplitudes are used. The parameters $\alpha=1$, $\beta=2$, $\gamma_1=\gamma_3=c=1$, $\gamma_1=-1$, and $k=5$ are set in this case. Four cases of fractal wave solutions with $y=5$ are shown in fig. 1, where fig. 1(a): $p=q=r=1$, fig. 1(b) $p=0.8$, $q=0.7$, $r=0.6$, fig. 1(c) $p=0.6$, $q=0.5$, $r=0.4$, and fig. 1(d) $p=0.4$, $q=0.2$, $r=0.2$ are considered, respectively. One can see that the periodic propagation behavior appears when the conventional BK equation is considered. The oscillation fades away with the decrease of the fractal dimensions, see figs. 1(c) and 1(d). We also test the impact of the amplitude, A, about the fractal wave solutions with fixed $x=y=1$. Figure 2 provides two graphs with two different amplitudes, A, where $A=10$ and $A=15$ are set for the left and right sides of fig. 2, respectively. The variation of the function u about the fractal dimensions is similar to the numerical behavior in fig. 1. We can find that the variation of u gradually slows down as the amplitude, A, increases largely. Sensitivity analysis of the frequency, ω, with respect to the amplitude, A, is also considered, and the numerical curve is given in fig. 3. Figure 3 indicates the monotonic decreasing property of the frequency about the amplitude, A.

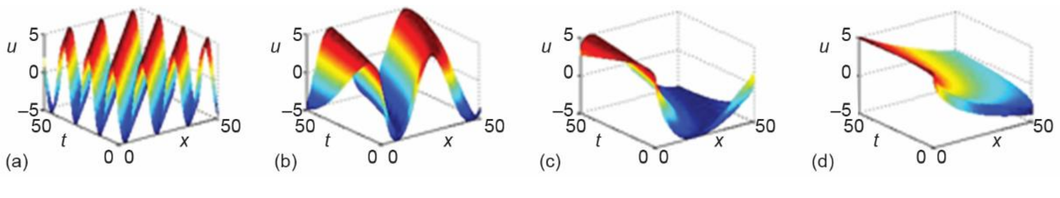

Figure 1. Propagation behavior of fractal periodic wave solutions to eq. (3) with $0\le x, t\le 50$

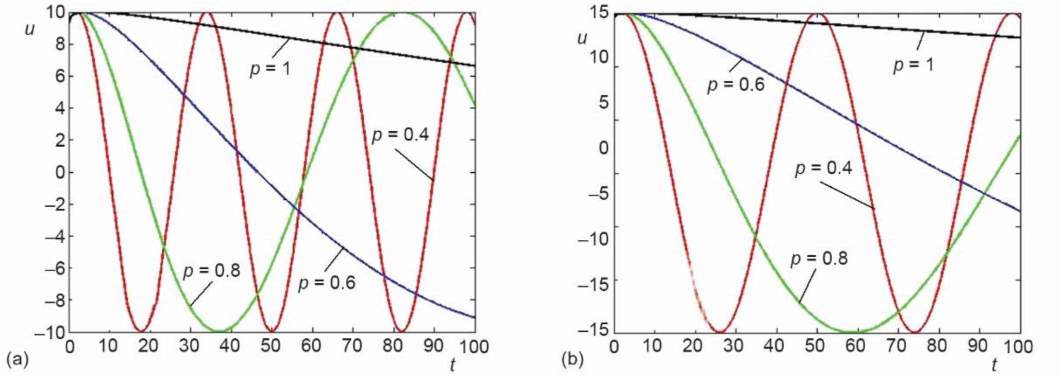

Figure 2. Curves of fractal periodic wave solutions; (a) A = 10 and (b) A = 15

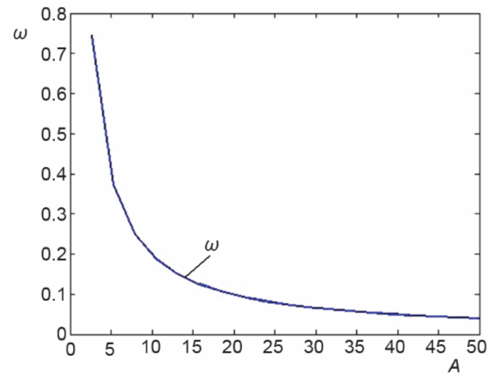

Figure 3. Curve of the frequency ω with $0 < A \le 50$

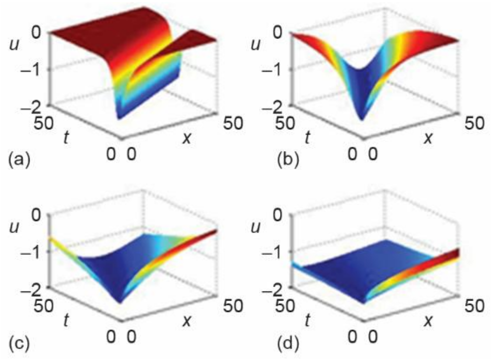

Figure 4. Numerical behavior of fractal briton soliton solutions to (3) with $0 \le x, t \le 50$

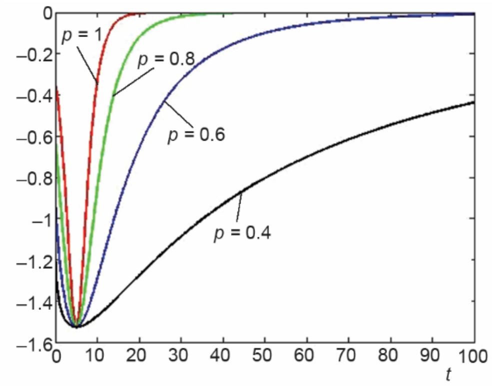

Figure 5. Curves of fractal briton soliton solutions with different fractal dimensions

The numerical results of the fractal briton soliton solutions to (3) are shown in fig. 4. The parameters are the same as those used in the previous fractal wave solutions, except c = 2. Four cases with different fractal dimensions p, q, and r are presented in figs. 4(a)-4(d), respectively. The propagation behavior of fractal solutions becomes smoother as the fractal order decreases. Numerical curves of the fractal soliton solutions with fixed $x = y = 5$ are given in fig. 5. The fluctuations of the fractal soliton solutions with respect to the fractal dimension are similar to the previous case.

Conclusion

In this work, we focused on the numerical investigation of the fractal gBK equation defined by He's fractal derivative. With the aid of the two-scale fractal transformation and the variational approach, the fractal periodic wave solution and the fractal briton soliton solution to the fractal gBK equation were given. Correction results to the variational formulation in the references [29, 30] were given. We provided the numerical results of the fractal wave solutions with various fractal dimensions and amplitudes for understanding the propagation behavior. The sensitivity of the frequency about the amplitude was also considered in detail. The fractal soliton solution can been seen as the corrected solution in the existing works [29, 30]. The variational approach is efficient for solving the fractal gBK equation, and our future work will extend this technique to other fractal or fractional non-linear partial differential equations.

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