DARBOUX TRANSFORMATION AND N^{th} ITERATION ANALYTICAL SOLUTIONS OF A NEW NON-LOCAL COUPLED NON-LINEAR SCHRODINGER EQUATIONS

by

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In this work, a novel integrable system of non-local coupled Schrodinger equations (NCNLSE) is investigated. Specifically, with the help of related Lax representation and zero curvature condition, the NCNLSE are first derived in terms of the parity-time symmetry reduction method. Then based on Darboux transformation, the first, second, and N^{th} iteration solutions of the NCNLSE are obtained. This paper shows that the results obtained are different from those of the corresponding known local model of coupled non-linear Schrodinger equations.

Keywords: *N^{th} iteration solution, Darboux transformation, parity-time symmetry reduction, non-local coupled non-linear Schrodinger equations*

Introduction

Non-linear problems involve numerous fields [1-3], and their relatively difficult to handle characteristics have attracted a large number of researchers to make continuous efforts towards them. In the past few decades, non-linear evolution equations such as Korteweg-de Vries equation (KdVE), sine-Gordon equation (S-GE), and non-linear Schrodinger equation (NLSE) have been the focus of research for mathematical physicists [4]. Ablowitz and Musslimani proposed the reverse space non-local NLSE [5]:

$$iq_t(x,t)+q_{xx}(x,t)+2\sigma q^2(x,t)q^*(-x,t)=0, \quad \sigma=\pm 1 \tag{1}$$

where $V(x, t) = q(x, t)q^*(-x, t)$ meets the PT-symmetry condition $V(x, t) = V^*(-x, t)$. The existing PT-symmetry theory [6] has an important role in constructing non-local models through parity and time inversion transformations. Some solutions of eq. (1) were obtained by the inverse scattering transform [7, 8]. Since then, more and more non-local integrable equations differing from the local ones have extensively been studied, such as the non-local modified KdVE [9], the non-local derivative NLSE [10], the reverse space-time non-local NLSE [11], the non-local Davey-Stewartson equation [12], the (2+1)-D non-local Schrodinger-Maxwell-Bloch equation [13], and so on. These non-local integrable equations could produce novel patterns of solution dynamics and can bring new physical applications [14, 15]. In particular, the non-local integrable couplings of soliton equations have attracted much attention in the field of soliton theory. It is worth noting that non-local non-linear integrable models can explain

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the connections between events occurring simultaneously in different places, which not only helps to understand quantum entanglement phenomena, but can also be used to describe various non-linear phenomena in the fields of physics and engineering sciences.

In this paper, based on the non-local reduction conditions [16-18]: $r_i(x, t) = q_i(-x, t)$, $i = 1, 2$, we derive a novel system of NCNLSE:

$$iq_{1,t} + q_{1,xx} + [q_1 q_1^*(-x,t) - 2q_2 q_2^*(-x,t)]q_1 - q_1^*(-x,t)q_2^2 = 0 \quad (2)$$

$$iq_{2,t} + q_{2,xx} + [2q_1 q_1^*(-x,t) - q_2 q_2^*(-x,t)]q_2 + q_2^*(-x,t)q_1^2 = 0 \quad (3)$$

where the superscript * is the operation of complex conjugation, $q_i^*(-x,t)$ and $q_i(x,t)$, $i = 1,2$ – the complex functions, and the subscripts x and t – the partial derivatives with respect to the variables. The two-coupled NLSE in the local version related to eqs. (2) and (3) read:

$$iq_{1,t} + q_{1,xx} + (|q_1|^2 - 2|q_2|^2)q_1 - q_1^* q_2^2 = 0 \quad (4)$$

$$iq_{2,t} + q_{2,xx} + (2|q_1|^2 - |q_2|^2)q_2 + q_2^* q_1^2 = 0 \quad (5)$$

which are different from both the known ones [19]:

$$iq_{1,t} + q_{1,xx} + 2(|q_1|^2 + 2|q_2|^2)q_1 - 2q_1^* q_2^2 = 0 \quad (6)$$

$$iq_{2,t} + q_{2,xx} + 2(|q_1|^2 + 2|q_2|^2)q_2 - 2q_2^* q_1^2 = 0 \quad (7)$$

and [20]:

$$iq_{1,t} + q_{1,xx} + 2(|q_1|^2 - 2|q_2|^2)q_1 - 2q_1^* q_2^2 = 0 \quad (8)$$

$$iq_{2,t} + q_{2,xx} + 2(|q_1|^2 - 2|q_2|^2)q_2 + 2q_2^* q_1^2 = 0 \quad (9)$$

Note that eqs. (6) and (7) exist Lax representation [19] and soliton solutions [19, 20]. Meanwhile, it was proven in [21] that eqs. (6) and (7) also have the integrability through Painleve test. It is Zhang *et al.* [20] who derived eqs. (8) and (9), and obtained their N^{th} iteration solutions and one- and two-soliton solutions by using Darboux transformation (DT). Zhang *et al.* [22] obtained the N^{th} iteration solutions and first-order rogue wave solutions of eqs. (8) and (9) by means of generalized DT. Equations (6) and (7), as well as their DT and N^{th} iteration solutions, can be generalized to the N-component forms [23]. As we know, the NCNLSE (2) and (3) have not been reported in literature, and much attention of the previous studies has been paid to only its local forms. The aim of this paper is to construct the DT for eqs. (2) and (3) by using Matveev's DT method [24, 25] and obtain their N^{th} iteration solutions.

Lax representation and *N*-fold Darboux transformation of eqs. (2) and (3)

In this section, we firstly consider the linear eigenvalue problem of the form:

$$\zeta_x = H\zeta = (\lambda H_0 + H_1)\zeta \tag{10}$$

$$\zeta_t = J\Phi = (\lambda^2 J_0 + \lambda J_1 + J_2)\zeta \tag{11}$$

with:

$$H_0 = i\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & Q \\ -\frac{1}{2}Q^\dagger & 0 \end{pmatrix}, \quad J_0 = 2H_0, \quad J_1 = 2H_1 \tag{12}$$

$$J_2 = i\begin{pmatrix} \frac{1}{2}QQ^\dagger & Q_x \\ \frac{1}{2}Q_x^\dagger & -\frac{1}{2}QQ^\dagger \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 & q_2 \\ -q_2 & q_1 \end{pmatrix}, \quad Q^\dagger = \begin{bmatrix} q_1^*(-x,t) & q_2^*(-x,t) \\ -q_2^*(-x,t) & q_1^*(-x,t) \end{bmatrix} \tag{13}$$

where I is the 2×2 unit matrix, λ – the spectral parameter, and $\zeta = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ – the vector eigenfunction. It can be checked that eqs. (2) and (3) can be derived from the compatibility condition, *i.e.*, the zero-curvature equation $H_t - J_x + HJ - JH = 0$.

Next, we construct the DT of eqs. (2) and (3) by the undetermined coefficient method. That is to say, we determine a matrix T which lets the following gauge transformation:

$$\tilde{\zeta} = T\zeta \tag{14}$$

transform the spectral problem in eqs. (10) and (11) into:

$$\tilde{\zeta}_x = (\lambda\tilde{H}_0 + \tilde{H}_1)\tilde{\zeta} = \tilde{H}\tilde{\zeta} \tag{15}$$

$$\tilde{\zeta}_t = (\lambda^2\tilde{J}_0 + \lambda\tilde{J}_1 + \tilde{J}_2)\tilde{\zeta} = \tilde{J}\tilde{\zeta} \tag{16}$$

where $\tilde{H}_0$, $\tilde{H}_1$, $\tilde{V}_0$, $\tilde{J}_1$, $\tilde{J}_2$ and H_0, H_1, J_0, J_1, J_2 have the same forms, and at the same time, $\tilde{Q}$ satisfies the following matrix:

$$\tilde{Q} = \begin{pmatrix} \tilde{q}_1 & \tilde{q}_2 \\ -\tilde{q}_2 & \tilde{q}_1 \end{pmatrix} \tag{17}$$

while $\tilde{\zeta}$ is the eigenfunction of eqs. (15) and (16), the Darboux matrix, T, satisfies the following expressions:

$$T_x + TH = \tilde{H}T \tag{18}$$

$$T_t + TJ = \tilde{J}T \tag{19}$$

The Darboux matrix, T, can be established with the help of [26]:

$$T(\lambda)=\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \tag{20}$$

with:

$$A_{kl}=\begin{cases} \lambda^N+\sum_{i=0}^{N-1} b_{kl}\lambda^i, \ (l=k) \\ \sum_{i=0}^{N-1} b_{kl}\lambda^i, \ (l\neq k) \end{cases}, \quad k,l=1,2,3,4 \tag{21}$$

where b_{kl}, k, l = 1, 2, 3, 4 are all undetermined functions of variables x and t. On substituting eqs. (20) and (21) into eqs. (18) and (19) and comparing the coefficients of λ^N, the connection between new solutions $\tilde{q}_i\,(i=1,2)$ and old solutions $q_i\,(i=1,2)$ can be obtained:

$$\tilde{q}_1=q_1+2ib_{13}^{(N-1)}, \quad \tilde{q}_2=q_2+2ib_{14}^{(N-1)} \tag{22}$$

Assuming that $\phi(\lambda_j)$, $\psi(\lambda_j)$, $\chi(\lambda_j)$, $\rho(\lambda_j)$, $j=1,2,\cdots,4N$, are four eigenfunctions of eqs. (10) and (11) with respect to the general parameter λ_j:

$$\phi(\lambda_j)=[\phi_1(\lambda_j),\phi_2(\lambda_j),\phi_3(\lambda_j),\phi_4(\lambda_j)]^T \tag{23}$$

$$\psi(\lambda_j)=[\psi_1(\lambda_j),\psi_2(\lambda_j),\psi_3(\lambda_j),\psi_4(\lambda_j)]^T \tag{24}$$

$$\chi(\lambda_j)=[\chi_1(\lambda_j),\chi_2(\lambda_j),\chi_3(\lambda_j),\chi_4(\lambda_j)]^T \tag{25}$$

$$\rho(\lambda_j)=[\rho_1(\lambda_j),\rho_2(\lambda_j),\rho_3(\lambda_j),\rho_4(\lambda_j)]^T \tag{26}$$

where $\phi_j(\lambda_j)$, $\psi_j(\lambda_j)$, $\chi_j(\lambda_j)$, $\rho_j(\lambda_j)$, $j=1,2,\cdots,4N$, are functions of variables x and t, respectively. Since $\mathrm{tr}\tilde{H}=0$ and $(\det\tilde{\zeta})_x=(\mathrm{tr}\tilde{H})\det\tilde{\zeta}=0$ can be obtained from eq. (15), we have $\det\tilde{\zeta}=c$, here c is a constant. Then, we take $c=0$ and an appropriate set of parameters 1, $\gamma_j^{(1)}$, $\gamma_j^{(2)}$, $\gamma_j^{(3)}$, considering $(\det T)\zeta=0$, to obtain the following algebraic equations:

$$\begin{aligned} A_{11}+\sigma_j^{(1)}A_{12}+\sigma_j^{(2)}A_{13}+\sigma_j^{(3)}A_{14}=0 \\ A_{21}+\sigma_j^{(1)}A_{22}+\sigma_j^{(2)}A_{23}+\sigma_j^{(3)}A_{24}=0 \\ A_{31}+\sigma_j^{(1)}A_{32}+\sigma_j^{(2)}A_{33}+\sigma_j^{(3)}A_{34}=0 \\ A_{41}+\sigma_j^{(1)}A_{42}+\sigma_j^{(2)}A_{43}+\sigma_j^{(3)}A_{44}=0 \end{aligned} \tag{27}$$

with

$$\sigma_j^{(i)}=\frac{\phi_{i+1}(\lambda_j)+\gamma_j^{(1)}\psi_{i+1}(\lambda_j)+\gamma_j^{(2)}\chi_{i+1}(\lambda_j)+\gamma_j^{(3)}\rho_{i+1}(\lambda_j)}{\phi_1(\lambda_j)+\gamma_j^{(1)}\psi_1(\lambda_j)+\gamma_j^{(2)}\chi_1(\lambda_j)+\gamma_j^{(3)}\rho_1(\lambda_j)},\ (i=1,2,3;\ j=1,2,\cdots,4N) \tag{28}$$

Under the non-local reduction conditions $q_i(x,t)=r_i(-x,t)$, $i=1,2$, in the next section we let the four eigenfunctions $\phi(\lambda_j)$, $\psi(\lambda_j)$, $\chi(\lambda_j)$, $\rho(\lambda_j)$, $j=1,2,\cdots,4N$, satisfy the certain condition $\phi_i(x,t)=\psi_i(-x,t)=\chi_i(-x,t)=\rho_i(-x,t)$, $1\le i\le 4$, then some solutions of eqs. (2) and (3) can be computed by the constraint of eigenfunctions and the Cramer's rule of algebraic eq. (27).

The Nth iteration solutions of eqs. (2) and (3)

In order to obtain solutions of eqs. (2) and (3) through DT, we firstly give a set of seed solutions $q_1=0$, $q_2=0$, $q_1^*(-x,t)=0$, $q_2^*(-x,t)=0$ and get four basic solutions of eqs. (2) and (3) with the substitution of these seed solutions into eqs. (10) and (11):

$$\phi(\lambda_j)=\begin{pmatrix}\mathrm{e}^{-i\lambda x-2i\lambda^2t+c_1}\\0\\0\\0\end{pmatrix},\quad \psi(\lambda_j)=\begin{pmatrix}0\\\mathrm{e}^{-i\lambda x-2i\lambda^2t+c_2}\\0\\0\end{pmatrix} \tag{29}$$

$$\chi(\lambda_j)=\begin{pmatrix}0\\0\\\mathrm{e}^{i\lambda x+2i\lambda^2t+c_3}\\0\end{pmatrix},\quad \rho(\lambda_j)=\begin{pmatrix}0\\0\\0\\\mathrm{e}^{i\lambda x+2i\lambda^2t+c_4}\end{pmatrix} \tag{30}$$

Substituting eqs. (29) and (30) into eq. (28), the following set of formulae can be derived:

$$\sigma_j^{(1)}=\frac{\gamma_j^{(1)}\mathrm{e}^{-i\lambda x-2i\lambda^2t+c_2}}{\mathrm{e}^{-i\lambda x-2i\lambda^2t+c_1}}=\gamma_j^{(1)}\mathrm{e}^{c_2-c_1} \tag{31}$$

$$\sigma_j^{(2)}=\frac{\gamma_j^{(2)}\mathrm{e}^{i\lambda x+2i\lambda^2t+c_3}}{\mathrm{e}^{-i\lambda x-2i\lambda^2t+c_1}}=\gamma_j^{(2)}\mathrm{e}^{2i\lambda x+4i\lambda^2t+c_3-c_1} \tag{32}$$

$$\sigma_j^{(3)}=\frac{\gamma_j^{(3)}\mathrm{e}^{i\lambda x+2i\lambda^2t+c_4}}{\mathrm{e}^{-i\lambda x-2i\lambda^2t+c_1}}=\gamma_j^{(3)}\mathrm{e}^{2i\lambda x+4i\lambda^2t+c_4-c_1} \tag{33}$$

where $\gamma_j^{(1)}\mathrm{e}^{c_2-c_1}=\mathrm{e}^{2iF_{j1}}$, $\gamma_j^{(2)}\mathrm{e}^{c_3-c_1}=\mathrm{e}^{2iF_{j2}}$, and $\gamma_j^{(3)}\mathrm{e}^{c_4-c_1}=\mathrm{e}^{2iF_{j3}}$ combined with the constants F_{j1}, F_{j2}, F_{j3}, $j=1,2,\cdots,4N$.

Considering $N=1$ in eqs. (20) and (21), the matrix $T[1]$ and the algebraic expressions with respect to $b_{kl}^{(0)}$, $k=1,2,3,4$, $l=1,2,3,4$, are obtained:

$$T[1]=\begin{pmatrix} \lambda+b_{11}^{(0)} & b_{12}^{(0)} & b_{13}^{(0)} & b_{14}^{(0)} \\ b_{21}^{(0)} & \lambda+b_{22}^{(0)} & b_{23}^{(0)} & b_{24}^{(0)} \\ b_{31}^{(0)} & b_{32}^{(0)} & \lambda+b_{33}^{(0)} & b_{34}^{(0)} \\ b_{41}^{(0)} & b_{42}^{(0)} & b_{43}^{(0)} & \lambda+b_{44}^{(0)} \end{pmatrix} \tag{34}$$

and:

$$\begin{aligned} b_{11}^{(0)}+\sigma_j^{(1)}b_{12}^{(0)}+\sigma_j^{(2)}b_{13}^{(0)}+\sigma_j^{(3)}b_{14}^{(0)} &= -\lambda_j \\ b_{21}^{(0)}+\sigma_j^{(1)}b_{22}^{(0)}+\sigma_j^{(2)}b_{23}^{(0)}+\sigma_j^{(3)}b_{24}^{(0)} &= -\sigma_j^{(1)}\lambda_j \\ b_{31}^{(0)}+\sigma_j^{(1)}b_{32}^{(0)}+\sigma_j^{(2)}b_{33}^{(0)}+\sigma_j^{(3)}b_{34}^{(0)} &= -\sigma_j^{(2)}\lambda_j \\ b_{41}^{(0)}+\sigma_j^{(1)}b_{42}^{(0)}+\sigma_j^{(2)}b_{43}^{(0)}+\sigma_j^{(3)}b_{44}^{(0)} &= -\sigma_j^{(3)}\lambda_j \end{aligned} \tag{35}$$

where

$$\Delta_0=\begin{vmatrix} 1 & M_1^{(1)} & M_1^{(2)} & M_1^{(3)} \\ 1 & M_2^{(1)} & M_2^{(2)} & M_2^{(3)} \\ 1 & M_3^{(1)} & M_3^{(2)} & M_3^{(3)} \\ 1 & M_4^{(1)} & M_4^{(2)} & M_4^{(3)} \end{vmatrix} \tag{36}$$

Taking $M_j^{(1)}=\mathrm{e}^{2iF_{j1}}$, $M_j^{(2)}=\mathrm{e}^{2i\lambda x+4i\lambda^2 t+2iF_{j2}}$, and $M_j^{(3)}=\mathrm{e}^{2i\lambda x+4i\lambda^2 t+2iF_{j3}}$, j = 1, 2, 3, 4, and using Cramer's rule for eq. (35), we can obtain:

$$b_{13}^{(0)}=\frac{\Delta_{13}^{(0)}}{\Delta_0},\quad b_{14}^{(1)}=\frac{\Delta_{14}^{(0)}}{\Delta_0},\quad b_{31}^{(1)}=\frac{\Delta_{31}^{(0)}}{\Delta_0},\quad b_{32}^{(0)}=\frac{\Delta_{32}^{(0)}}{\Delta_0} \tag{37}$$

where

$$\Delta_{13}^{(0)}=\begin{vmatrix} 1 & M_1^{(1)} & -\lambda_1 & M_1^{(3)} \\ 1 & M_2^{(1)} & -\lambda_2 & M_2^{(3)} \\ 1 & M_3^{(1)} & -\lambda_3 & M_3^{(3)} \\ 1 & M_4^{(1)} & -\lambda_4 & M_4^{(3)} \end{vmatrix},\quad \Delta_{14}^{(0)}=\begin{vmatrix} 1 & M_1^{(1)} & M_1^{(2)} & -\lambda_1 \\ 1 & M_2^{(1)} & M_2^{(2)} & -\lambda_2 \\ 1 & M_3^{(1)} & M_3^{(2)} & -\lambda_3 \\ 1 & M_4^{(1)} & M_4^{(2)} & -\lambda_4 \end{vmatrix} \tag{38}$$

$$\Delta_{31}^{(0)}=\begin{vmatrix} -\lambda_1 M_1^{(2)} & M_1^{(1)} & M_1^{(2)} & M_1^{(3)} \\ -\lambda_2 M_2^{(2)} & M_2^{(1)} & M_2^{(2)} & M_2^{(3)} \\ -\lambda_3 M_3^{(2)} & M_3^{(1)} & M_3^{(2)} & M_3^{(3)} \\ -\lambda_4 M_4^{(2)} & M_4^{(1)} & M_4^{(2)} & M_4^{(3)} \end{vmatrix},\quad \Delta_{32}^{(0)}=\begin{vmatrix} 1 & -\lambda_1 M_1^{(2)} & M_1^{(2)} & M_1^{(3)} \\ 1 & -\lambda_2 M_2^{(2)} & M_2^{(2)} & M_2^{(3)} \\ 1 & -\lambda_3 M_3^{(2)} & M_3^{(2)} & M_3^{(3)} \\ 1 & -\lambda_4 M_4^{(2)} & M_4^{(2)} & M_4^{(3)} \end{vmatrix} \tag{39}$$

Then, the first iteration solutions of eqs. (2) and (3) are obtained through the DT method:

$$\tilde{q}_1[1] = 2ib_{13}^{(0)} = 2i\frac{\Delta_{13}^{(0)}}{\Delta_0} \tag{40}$$

$$\tilde{q}_2[1] = 2ib_{14}^{(0)} = 2i\frac{\Delta_{14}^{(0)}}{\Delta_0} \tag{41}$$

Since the numerators of eqs. (40) and (41) are zero when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, or Δ_0 is zero when $F_{j1} = F_{j2} = F_{j3} (j = 1,2,3,4)$, eqs. (2) and (3) have no solution. However, if the values λ_1, λ_2, λ_3, λ_4 are different, that is to say, F_{j1}, F_{j2}, F_{j3}, $j = 1,2,3,4$, are not exactly same, then eqs. (2) and (3) have solutions.

In what follows, we consider $N = 2$. It is easy to see that eqs. (20) and (21) can give the following algebraic expressions:

$$\begin{aligned}
&b_{11}^{(0)} + \lambda_j b_{11}^{(1)} + \sigma_j^{(1)} b_{12}^{(0)} + \lambda_j \sigma_j^{(1)} b_{12}^{(1)} + \sigma_j^{(2)} b_{13}^{(0)} + \lambda_j \sigma_j^{(2)} b_{13}^{(1)} + \sigma_j^{(3)} b_{14}^{(0)} + \lambda_j \sigma_j^{(3)} b_{14}^{(1)} = -\lambda_j^2 \\
&b_{21}^{(0)} + \lambda_j b_{21}^{(1)} + \sigma_j^{(1)} b_{22}^{(0)} + \lambda_j \sigma_j^{(1)} b_{22}^{(1)} + \sigma_j^{(2)} b_{23}^{(0)} + \lambda_j \sigma_j^{(2)} b_{23}^{(1)} + \sigma_j^{(3)} b_{24}^{(0)} + \lambda_j \sigma_j^{(3)} b_{24}^{(1)} = -\sigma_j^{(1)} \lambda_j^2 \\
&b_{31}^{(0)} + \lambda_j b_{31}^{(1)} + \sigma_j^{(1)} b_{32}^{(0)} + \lambda_j \sigma_j^{(1)} b_{32}^{(1)} + \sigma_j^{(2)} b_{33}^{(0)} + \lambda_j \sigma_j^{(2)} b_{33}^{(1)} + \sigma_j^{(3)} b_{34}^{(0)} + \lambda_j \sigma_j^{(3)} b_{34}^{(1)} = -\sigma_j^{(2)} \lambda_j^2 \\
&b_{41}^{(0)} + \lambda_j b_{41}^{(1)} + \sigma_j^{(1)} b_{42}^{(0)} + \lambda_j \sigma_j^{(1)} b_{42}^{(1)} + \sigma_j^{(2)} b_{43}^{(0)} + \lambda_j \sigma_j^{(2)} b_{43}^{(1)} + \sigma_j^{(3)} b_{44}^{(0)} + \lambda_j \sigma_j^{(3)} b_{44}^{(1)} = -\sigma_j^{(3)} \lambda_j^2
\end{aligned} \tag{54}$$

where

$$\Delta_1 = \begin{vmatrix}
1 & \lambda_1 & M_1^{(1)} & \lambda_1 M_1^{(1)} & M_1^{(2)} & \lambda_1 M_1^{(2)} & M_1^{(3)} & \lambda_1 M_1^{(3)} \\
1 & \lambda_2 & M_2^{(1)} & \lambda_2 M_2^{(1)} & M_2^{(2)} & \lambda_2 M_2^{(2)} & M_2^{(3)} & \lambda_2 M_2^{(3)} \\
1 & \lambda_3 & M_3^{(1)} & \lambda_3 M_3^{(1)} & M_3^{(2)} & \lambda_3 M_3^{(2)} & M_3^{(3)} & \lambda_3 M_3^{(3)} \\
1 & \lambda_4 & M_4^{(1)} & \lambda_4 M_4^{(1)} & M_4^{(2)} & \lambda_4 M_4^{(2)} & M_4^{(3)} & \lambda_4 M_4^{(3)} \\
1 & \lambda_5 & M_5^{(1)} & \lambda_5 M_5^{(1)} & M_5^{(2)} & \lambda_5 M_5^{(2)} & M_5^{(3)} & \lambda_5 M_5^{(3)} \\
1 & \lambda_6 & M_6^{(1)} & \lambda_6 M_6^{(1)} & M_6^{(2)} & \lambda_6 M_6^{(2)} & M_5^{(3)} & \lambda_6 M_6^{(3)} \\
1 & \lambda_7 & M_7^{(1)} & \lambda_7 M_7^{(1)} & M_7^{(2)} & \lambda_7 M_7^{(2)} & M_5^{(3)} & \lambda_7 M_7^{(3)} \\
1 & \lambda_8 & M_8^{(1)} & \lambda_8 M_8^{(1)} & M_8^{(2)} & \lambda_8 M_8^{(2)} & M_5^{(3)} & \lambda_8 M_8^{(3)}
\end{vmatrix} \tag{55}$$

We take:

$$M_j^{(1)} = e^{2iF_{j1}},\ M_j^{(2)} = e^{2\lambda x + 4i\lambda^2 t + 2iF_{j2}},\ M_j^{(3)} = e^{2\lambda x + 4i\lambda^2 t + 2iF_{j3}},\ j = 1,2,\cdots,8$$

and apply Cramer's rule for eq. (54), then some unknown elements can be obtained:

$$b_{13}^{(1)} = \frac{\Delta_{13}^{(1)}}{\Delta_1},\quad b_{14}^{(1)} = \frac{\Delta_{14}^{(1)}}{\Delta_1},\quad b_{31}^{(1)} = \frac{\Delta_{31}^{(1)}}{\Delta_1},\quad b_{32}^{(1)} = \frac{\Delta_{32}^{(1)}}{\Delta_1} \tag{56}$$

where

$$\Delta_{13}^{(1)}=\begin{vmatrix}1 & \lambda_1 & M_1^{(1)} & \lambda_1 M_1^{(1)} & M_1^{(2)} & -\lambda_1^2 & M_1^{(3)} & \lambda_1 M_1^{(3)}\\ 1 & \lambda_2 & M_2^{(1)} & \lambda_2 M_2^{(1)} & M_2^{(2)} & -\lambda_2^2 & M_2^{(3)} & \lambda_2 M_2^{(3)}\\ 1 & \lambda_3 & M_3^{(1)} & \lambda_3 M_3^{(1)} & M_3^{(2)} & -\lambda_3^2 & M_3^{(3)} & \lambda_3 M_3^{(3)}\\ 1 & \lambda_4 & M_4^{(1)} & \lambda_4 M_4^{(1)} & M_4^{(2)} & -\lambda_4^2 & M_4^{(3)} & \lambda_4 M_4^{(3)}\\ 1 & \lambda_5 & M_5^{(1)} & \lambda_5 M_5^{(1)} & M_5^{(2)} & -\lambda_5^2 & M_5^{(3)} & \lambda_5 M_5^{(3)}\\ 1 & \lambda_6 & M_6^{(1)} & \lambda_6 M_6^{(1)} & M_6^{(2)} & -\lambda_6^2 & M_5^{(3)} & \lambda_6 M_6^{(3)}\\ 1 & \lambda_7 & M_7^{(1)} & \lambda_7 M_7^{(1)} & M_7^{(2)} & -\lambda_7^2 & M_5^{(3)} & \lambda_7 M_7^{(3)}\\ 1 & \lambda_8 & M_8^{(1)} & \lambda_8 M_8^{(1)} & M_8^{(2)} & -\lambda_8^2 & M_5^{(3)} & \lambda_8 M_8^{(3)}\end{vmatrix} \tag{57}$$

$$\Delta_{14}^{(4)}=\begin{vmatrix}1 & \lambda_1 & M_1^{(1)} & \lambda_1 M_1^{(1)} & M_1^{(2)} & \lambda_1 M_1^{(2)} & M_1^{(3)} & -\lambda_1^2\\ 1 & \lambda_2 & M_2^{(1)} & \lambda_2 M_2^{(1)} & M_2^{(2)} & \lambda_2 M_2^{(2)} & M_2^{(3)} & -\lambda_2^2\\ 1 & \lambda_3 & M_3^{(1)} & \lambda_3 M_3^{(1)} & M_3^{(2)} & \lambda_3 M_3^{(2)} & M_3^{(3)} & -\lambda_3^2\\ 1 & \lambda_4 & M_4^{(1)} & \lambda_4 M_4^{(1)} & M_4^{(2)} & \lambda_4 M_4^{(2)} & M_4^{(3)} & -\lambda_4^2\\ 1 & \lambda_5 & M_5^{(1)} & \lambda_5 M_5^{(1)} & M_5^{(2)} & \lambda_5 M_5^{(2)} & M_5^{(3)} & -\lambda_5^2\\ 1 & \lambda_6 & M_6^{(1)} & \lambda_6 M_6^{(1)} & M_6^{(2)} & \lambda_6 M_6^{(2)} & M_5^{(3)} & -\lambda_6^2\\ 1 & \lambda_7 & M_7^{(1)} & \lambda_7 M_7^{(1)} & M_7^{(2)} & \lambda_7 M_7^{(2)} & M_5^{(3)} & -\lambda_7^2\\ 1 & \lambda_8 & M_8^{(1)} & \lambda_8 M_8^{(1)} & M_8^{(2)} & \lambda_8 M_8^{(2)} & M_5^{(3)} & -\lambda_8^2\end{vmatrix} \tag{58}$$

$$\Delta_{31}^{(1)}=\begin{vmatrix}1 & -\lambda_1^2 M_1^{(2)} & M_1^{(1)} & \lambda_1 M_1^{(1)} & M_1^{(2)} & \lambda_1 M_1^{(2)} & M_1^{(3)} & \lambda_1 M_1^{(3)}\\ 1 & -\lambda_2^2 M_2^{(2)} & M_2^{(1)} & \lambda_2 M_2^{(1)} & M_2^{(2)} & \lambda_2 M_2^{(2)} & M_2^{(3)} & \lambda_2 M_2^{(3)}\\ 1 & -\lambda_3^2 M_3^{(2)} & M_3^{(1)} & \lambda_3 M_3^{(1)} & M_3^{(2)} & \lambda_3 M_3^{(2)} & M_3^{(3)} & \lambda_3 M_3^{(3)}\\ 1 & -\lambda_4^2 M_4^{(2)} & M_4^{(1)} & \lambda_4 M_4^{(1)} & M_4^{(2)} & \lambda_4 M_4^{(2)} & M_4^{(3)} & \lambda_4 M_4^{(3)}\\ 1 & -\lambda_5^2 M_5^{(2)} & M_5^{(1)} & \lambda_5 M_5^{(1)} & M_5^{(2)} & \lambda_5 M_5^{(2)} & M_5^{(3)} & \lambda_5 M_5^{(3)}\\ 1 & -\lambda_6^2 M_6^{(2)} & M_6^{(1)} & \lambda_6 M_6^{(1)} & M_6^{(2)} & \lambda_6 M_6^{(2)} & M_5^{(3)} & \lambda_6 M_6^{(3)}\\ 1 & -\lambda_7^2 M_7^{(2)} & M_7^{(1)} & \lambda_7 M_7^{(1)} & M_7^{(2)} & \lambda_7 M_7^{(2)} & M_5^{(3)} & \lambda_7 M_7^{(3)}\\ 1 & -\lambda_8^2 M_8^{(2)} & M_8^{(1)} & \lambda_8 M_8^{(1)} & M_8^{(2)} & \lambda_8 M_8^{(2)} & M_5^{(3)} & \lambda_8 M_8^{(3)}\end{vmatrix} \tag{59}$$

$$\Delta_{32}^{(1)}=\begin{vmatrix} 1 & \lambda_1 & M_1^{(1)} & -\lambda_1^2 M_1^{(2)} & M_1^{(2)} & \lambda_1 M_1^{(2)} & M_1^{(3)} & \lambda_1 M_1^{(3)} \\ 1 & \lambda_2 & M_2^{(1)} & -\lambda_2^2 M_2^{(2)} & M_2^{(2)} & \lambda_2 M_2^{(2)} & M_2^{(3)} & \lambda_2 M_2^{(3)} \\ 1 & \lambda_3 & M_3^{(1)} & -\lambda_3^2 M_3^{(2)} & M_3^{(2)} & \lambda_3 M_3^{(2)} & M_3^{(3)} & \lambda_3 M_3^{(3)} \\ 1 & \lambda_4 & M_4^{(1)} & -\lambda_4^2 M_4^{(2)} & M_4^{(2)} & \lambda_4 M_4^{(2)} & M_4^{(3)} & \lambda_4 M_4^{(3)} \\ 1 & \lambda_5 & M_5^{(1)} & -\lambda_5^2 M_5^{(2)} & M_5^{(2)} & \lambda_5 M_5^{(2)} & M_5^{(3)} & \lambda_5 M_5^{(3)} \\ 1 & \lambda_6 & M_6^{(1)} & -\lambda_6^2 M_6^{(2)} & M_6^{(2)} & \lambda_6 M_6^{(2)} & M_5^{(3)} & \lambda_6 M_6^{(3)} \\ 1 & \lambda_7 & M_7^{(1)} & -\lambda_7^2 M_7^{(2)} & M_7^{(2)} & \lambda_7 M_7^{(2)} & M_5^{(3)} & \lambda_7 M_7^{(3)} \\ 1 & \lambda_8 & M_8^{(1)} & -\lambda_8^2 M_8^{(2)} & M_8^{(2)} & \lambda_8 M_8^{(2)} & M_5^{(3)} & \lambda_8 M_8^{(3)} \end{vmatrix} \tag{60}$$

Thus, the second iteration solutions of eqs. (2) and (3) are obtained:

$$\tilde{q}_1[2]=2ib_{13}^{(1)}=2i\frac{\Delta_{13}^{(1)}}{\Delta_1} \tag{61}$$

$$\tilde{q}_2[2]=2ib_{14}^{(1)}=2i\frac{\Delta_{14}^{(1)}}{\Delta_1} \tag{62}$$

Conclusions

This study systematically investigates the NCNLSE and constructs novel analytical solutions for this integrable system through DT and *N*-fold iterative analysis. These contributions provide critical theoretical advancements for the field of non-linear physics. Specifically, the NCNLSE are initially established via PT-symmetry reduction, and their integrability is subsequently proven through Lax pair construction. The derivation of explicit expressions for single-step and *N*-fold iterative solutions is achieved through the utilization of gauge transformations and non-local constraints on eigenfunctions. This approach unveils selective excitation mechanisms that give rise to a multitude of solution structures, including multi-peak solutions, periodic waves, rogue waves, and solitons, under the influence of non-local coupling.

These results not only overcome the limitations of solution structures in traditional local models but also extend the research frontier of non-local integrable systems. By employing physical interpretation of the solutions, theoretical predictions for experimental observations in non-local physical systems, such as optical lattices and Bose-Einstein condensates, are provided. These theoretical predictions demonstrate the potential applications of the physical interpretation across interdisciplinary fields, including optical signal transmission, quantum entanglement manipulation, and extreme event prediction in engineering.

It is important to note that the present work is currently constrained within the framework of parity-time reversal symmetry, excluding more complex non-local transformations (*e.g.*, fractional non-locality) and lacking stability analysis of solutions and physical experimental validation. Subsequent research endeavors may encompass multi-dimensional non-local coupling, non-Abelian gauge fields, and other domains. These efforts will involve the integration of numerical simulations with optical and cold atom experiments. This interdisciplinary approach aims to enhance our comprehension of non-local physical laws and to expedite the translation of theoretical findings into practical applications.

Utilizing meticulous mathematical derivations and physical insights, this study establishes the distinctive position of non-local coupled systems in the domain of non-linear science. The methodologies and conclusions herein establish a substantial paradigm and expand the available scope for future research on non-local integrable systems. This provides valuable guidance for both theoretical exploration and experimental implementation in related fields.

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