

## TRANSMISSION DYNAMICS ANALYSIS OF A BISTABLE VAN DER POL OSCILLATOR UNDER GAUSSIAN COLORED NOISE WITH FRACTIONAL INERTIAL ELEMENT

by

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*The stochastic response and bifurcation phenomenon of system amplitude in a bi-stable Van der Pol system with fractional inertial element driven by the Gaussian colored noise is investigated. In the initial step of the process, the mean square error is minimized, and the harmonic balance technique is employed. This results in the fractional derivative being found to be isovalent to a linear combination of inertial and damping forces. Simultaneously, the original system is simplified to an equivalent integer order Van der Pol system. Secondly, the steady-state probability density function of the system amplitude is acquired via stochastic averaging. According to the principles of singularity theory, the critical parametric conditions for the stochastic P-bifurcation of system amplitude can be determined. A qualitative analysis of the stationary PDF curves of amplitude is finally conducted in each area, with the transition set curves serving as a dividing point. The congruence between the analytical outcomes and the numerical results obtained from Monte Carlo simulation and the radial basis function neural network testifies to the theoretical analysis. In this paper, the methodology employed and the results obtained can enhance the design of the fractional-order controller to control the response of these systems.*

Key words: *stochastic P-bifurcation, Gaussian colored noise, fractional damping, critical parametric conditions, Monte Carlo simulation*

### Introduction

Fractional calculus extends the scope of classical calculus to include non-integer orders, thereby facilitating the characterization of memory effects in viscoelastic substances with greater efficacy compared to integer-order derivatives. The definition of fractional derivative incorporates convolution, a mathematical operation that can express a memory effect and demonstrate a cumulative effect over time. Consequently, the fractional derivative emerges as a more suitable mathematical instrument for the analysis of memory characteristics

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[1-4]. It has become a powerful mathematical instrument for research in various fields, including anomalous diffusion, non-Newtonian fluid mechanics, viscoelastic mechanics, and soft matter physics. A comparison of the integer-order calculus and the fractional derivative reveals that the latter can describe various reaction processes with greater accuracy [5-7]. Consequently, it is necessary and significant to study the mechanical characteristics and the fractional order parametric influences on such systems.

In recent years, numerous scholars have investigated the dynamic behavior of non-linear multi-stable systems under various forms of noise excitations, yielding significant findings. Liu *et al.* [8] conducted a study of the response of a strongly non-linear vibro-impact system with Coulomb friction, excited by real noise. They then proceeded to analyze the P-bifurcation by means of a qualitative change in the friction amplitude and the restitution coefficient on the stationary probability distribution. A number of researchers [9-11] have examined the Duffing-Van der Pol oscillators in the context of Levy noise, colored noise, and a combination of harmonic and random noise, respectively. The stochastic P-bifurcation behaviors of the noise oscillators are examined through the analysis of alterations in the system stationary PDF. The analytical results of the bimodal stationary PDF demonstrate that system parameters and noise intensity can each induce stochastic P-bifurcation of the systems. Hao and Wu [12-14] investigated the tristable stochastic P-bifurcation in a generalized Duffing-Van der Pol oscillator under additive Gaussian white noise, multiplicative colored noise, combined additive and multiplicative Gaussian white noise, respectively. They obtained an analytical expression of the system stationary PDF of amplitude and analyzed the influences of noise intensity and system parameters on the system stochastic P-bifurcation. In their study, Chen and Zhu [15] examined the response of a Duffing system with fractional damping under the combined white noise and harmonic excitations. Their findings demonstrated that the variation of the fractional derivative order can induce the system stochastic P-bifurcation. Huang and Jin [16] explored the response and the stationary PDF of a single-degree-of-freedom strongly non-linear system under the influence of Gaussian white noise excitation. Li *et al.* [17] investigated the bi-stable stochastic P-bifurcation behavior of a Van der Pol-Duffing system with the fractional derivative under additive and multiplicative colored noise excitations. Their findings indicated that changes in the linear damping coefficient, the fractional derivative order, and the noise intensity can each result in stochastic P-bifurcation in the system. Liu *et al.* [18] investigated a Duffing oscillator system with fractional damping under combined harmonic and Poisson white noise parametric excitation. They then analyzed the asymptotic Lyapunov stability with probability of the original system based on the largest Lyapunov exponent. Chen *et al.* [19] conducted a study of the primary resonance response of a Van der Pol system under the influence of fractional-order delayed negative feedback and forced excitation. They employed the averaging method to derive an approximate analytical solution. Chen *et al.* [20] proposed a stochastic averaging technique for studying the behavior of a randomly excited strongly non-linear system with a delayed feedback fractional-order proportional-derivative controller. Through rigorous analysis, they were able to derive the stationary PDF of the system, thereby contributing to the advancement of the field. A considerable body of research has been dedicated to the study of non-linear vibration systems [21-29], with particular emphasis on those systems that have found extensive applications in energy harvesting and control. These include the six DoF system and the 3-DoF auto-parametric system.

The inherent complexity of the fractional derivative precludes the possibility of a quantitative analysis of the parametric vibration characteristics of the fractional system. Consequently, the critical conditions of the parametric influences remain unattainable. In practice,

the critical conditions of the parametric influences play a vital role in the analysis and design of fractional order systems. Furthermore, the stochastic P-bifurcation of bi-stability for the generalized Van der Pol system with the fractional inertial force has not been reported in the extant literature. In this paper, the non-linear vibrations of fractional order systems are studied through the fractional derivative. The study is based on a generalized Van der Pol system with fractional damping that is excited by multiplicative Gaussian white noise. The transition set curves and critical parameter conditions for the system stochastic P-bifurcation are obtained by the singularity method. An analysis of the system stationary PDF curves in each area of the parameter plane is conducted. A comparison is also made between the numerical results from Monte Carlo simulation and the analytical solutions obtained by stochastic averaging. The comparison demonstrates that the numerical results are in close agreement with the analytical solutions, thereby validating our theoretical analysis.

### Derivation of the isovalent system

The initial condition of the Riemann-Liouville derivative is devoid of physical meaning, while the initial condition of the system described by the Caputo derivative possesses both clear physical meaning and the same initial condition as the integer-order differential equation. Consequently, in this paper, we adopt the Caputo fractional derivative:

$$\begin{aligned}
 {}^c_a D^p[x(t)] &= \frac{1}{\Gamma(m-p)} \int_a^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} du, \quad m < p < m+1 \\
 {}^c_a D^p[x(t)] &= x^{(m)}(t), \quad p = m
 \end{aligned} \tag{1}$$

where  $m \in N$ ,  $t \in [a, b]$ ,  $x^{(m)}(t)$  is the  $m$ -order derivative of  $x(t)$  and  $\Gamma(m)$  – the Gamma function which satisfies  $\Gamma(m+1) = m\Gamma(m)$ .

For a given physical system, the initial moment of oscillators is  $t = 0$  and the Caputo derivative is usually expressed:

$${}^c_0 D^p[x(t)] = \frac{1}{\Gamma(m-p)} \int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}} du \tag{2}$$

where  $m-1 < p \leq m$ ,  $m \in N$ .

In this paper, we study the generalized Van der Pol system with the fractional damping excited by Gaussian colored noise excitation:

$${}^c_0 D^{1+p}[x(t)] - \mu[-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]\dot{x}(t) + w^2 x(t) = \eta(t) \tag{3}$$

which is isovalent to the following 3-D state equations:

$$\begin{aligned}
 {}^c_0 D^p[x(t)] &= y(t), \quad {}^c_0 D^{1-p}[y(t)] = z(t) \\
 \dot{y}(t) - \mu[-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]z(t) + w^2 x(t) &= \eta(t)
 \end{aligned} \tag{4}$$

where  $x(t)$  is the displacement of the system,  $\varepsilon$  – the linear damping coefficient,  $\alpha_1$  and  $\alpha_2$  – the non-linear damping coefficients of the system, and  $w$  – the system natural frequency. The  ${}^c_0 D^p[x(t)]$  is the  $p$  ( $0 \leq p \leq 1$ ) order Caputo derivative of  $x(t)$ , which is defined by eq. (2). In addition  $\xi(t)$  is a Gaussian White noise, whose statistical properties are determined by:

$$E[\xi(t)\xi(t+t')] = 2D\delta(t'), \quad E[\xi(t)] = 0$$

Setting

$$\dot{\eta}(t) = -\frac{1}{\tau}\eta(t) + \frac{1}{\tau}\zeta(t)$$

then it is called the colored noise, which satisfies:

$$E[\zeta(t_1)\zeta(t_2)] = \frac{D}{\tau} \exp\left[-\frac{|t_1 - t_2|}{\tau}\right] \quad (5)$$

The fractional derivative has the contributions of damping force and restoring force [30], hence, we introduce the equivalent system:

$$M(w, p)\ddot{x}(t) + C(w, p)\dot{x}(t) - u[-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]\dot{x}(t) + w^2 x(t) = \eta(t) \quad (6)$$

where  $C(p)$  and  $K(p)$  are the coefficients of the equivalent damping and equivalent restoring forces of the fractional derivative  ${}^C_0 D^p[x(t)]$ , respectively.

Applying the equivalent method mentioned in the reference [18], we get the ultimate forms of  $C(p)$  and  $K(p)$ :

$$M(p, w) = w^{p-1} \sin\left(\frac{p\pi}{2}\right), \quad C(p, w) = w^p \cos\left(\frac{p\pi}{2}\right) \quad (7)$$

Therefore, the isovalent Van der Pol oscillator associated with system (6) can be written:

$$\ddot{x}(t) - \mu_1 \gamma \dot{x} + w_0^2 x = \beta_1 \eta_1(t) \quad (8)$$

where

$$\gamma = -\varepsilon - \frac{w^p \cos\left(\frac{p\pi}{2}\right)}{\mu + \alpha_1 x^2 - \alpha_2 x^4} \quad (9)$$

$$\mu_1 = \frac{\mu w^{1-p}}{\sin\left(\frac{p\pi}{2}\right)}, \quad w_0^2 = \frac{w^{3-p}}{\sin\left(\frac{p\pi}{2}\right)}, \quad \beta_1 = \frac{w^{1-p}}{\sin\left(\frac{p\pi}{2}\right)}$$

### Steady PDF of the system amplitude

Assuming that system (8) has the solution of the periodic form, we introduce the following transformation [31]:

$$\begin{aligned} X &= x(t) = a(t) \cos \Phi(t) \\ Y &= \dot{x}(t) = -a(t)\Omega \sin \Phi(t) \\ \Phi(t) &= w_0 t + \theta(t) \end{aligned} \quad (10)$$

where  $w_0$  is natural frequency of the previous equivalent system (8),  $a(t)$  and  $\theta(t)$  –the amplitude and phase processes of the system response respectively, and they are both random processes.

Substituting eq. (10) into eq. (8), we obtain:

$$\begin{aligned} \frac{da}{dt} &= F_{11}(a, \theta) + \beta_1 G_{11}(a, \theta) \eta(t) \\ \frac{d\theta}{dt} &= F_{21}(a, \theta) + \beta_1 G_{21}(a, \theta) \eta(t) \end{aligned} \quad (11)$$

in which

$$\begin{aligned} F_{11}(a, \theta) &= u_1 \frac{\sin \Phi}{w_0} \left\{ aw_0 \sin \Phi \left[ -\varepsilon - \frac{w^p \cos\left(\frac{p\pi}{2}\right)}{\mu} + \alpha_1 a^2 \sin^2 \Phi - \alpha_2 a^4 \sin^4 \Phi \right] \right\} \\ F_{21}(a, \theta) &= u_1 \frac{\cos \Phi}{aw_0} \left\{ aw_0 \sin \Phi \left[ -\varepsilon - \frac{w^p \cos\left(\frac{p\pi}{2}\right)}{\mu} + \alpha_1 a^2 \sin^2 \Phi - \alpha_2 a^4 \sin^4 \Phi \right] \right\} \\ G_{11} &= -\frac{\sin \Phi}{w_0}, \quad G_{21} = -\frac{\cos \Phi}{aw_0} \end{aligned} \quad (12)$$

Equation (11) can be treated as the Stratonovich stochastic differential equation, and by adding the relevant Wong-Zakai correction term, we transform it into the corresponding Ito stochastic differential equation:

$$\begin{aligned} da &= [F_{11}(a, \theta) + F_{12}(a, \theta)]dt + \beta_1 G_{11}(a, \theta) \eta(t) \\ d\theta &= [F_{21}(a, \theta) + F_{22}(a, \theta)]dt + \beta_1 G_{21}(a, \theta) \eta(t) \end{aligned} \quad (13)$$

and

$$\begin{aligned} F_{12}(a, \theta) &= \frac{1}{2} \int_{-\infty}^{\infty} \left[ \frac{\partial G_{11}(a, \theta)}{\partial a} G_{11}(a, \theta, t+h) + \frac{\partial G_{11}(a, \theta)}{\partial \theta} G_{21}(a, \theta, t+h) \right] R(h) dh = \\ &= \frac{\beta_1^2 D \cos^2(\Phi)}{aw_0^2 (1 + w_0^2 \tau^2)} \\ F_{22}(a, \theta) &= \frac{1}{2} \int_{-\infty}^{\infty} \left[ \frac{\partial G_{21}(a, \theta)}{\partial a} G_{11}(a, \theta, t+h) + \frac{\partial G_{21}(a, \theta)}{\partial \theta} G_{21}(a, \theta, t+h) \right] R(h) dh = \\ &= -\frac{2\beta_1^2 D \sin(\Phi) \cos(\Phi)}{a^2 w_0^2 (1 + w_0^2 \tau^2)} \end{aligned} \quad (14)$$

in which:

$$R(h) = \frac{D}{\tau} \exp\left(-\frac{h}{\tau}\right)$$

Through stochastic averaging of eq. (13) over  $\Phi$  [32], we can obtain the following averaged Ito equation:

$$\begin{aligned} da &= m_1(a)dt + \sigma_{11}(a)dB(t) \\ d\theta &= m_2(a)dt + \sigma_{21}(a)dB(t) \end{aligned} \quad (15)$$

where

$$\begin{aligned}
 m_1(a) &= u_1 \left\{ -\frac{1}{2} \left[ \varepsilon + \frac{w^p \cos\left(\frac{p\pi}{2}\right)}{\mu} \right] a + \frac{1}{8} \alpha_1 a^3 - \frac{1}{16} \alpha_2 a^5 \right\} + \frac{\beta_1^2 D}{2aw_0^2(1+w_0^2\tau^2)} \\
 \sigma_{11}^2(a) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin^2(\Phi)}{w^2_0} S(w) d\Phi = \frac{\beta_1^2 D}{w_0^2(1+w_0^2\tau^2)} \\
 m_2(a) &= \frac{3u_1\beta_0 a^2}{8w_0} \\
 \sigma_{21}^2(a) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{2\cos^2(\Phi)}{a^2 w^2_0} S(w) d\Phi = \frac{\beta_1^2 D}{a^2 w_0^2(1+w_0^2\tau^2)}
 \end{aligned} \tag{16}$$

Equations (15) and (16) show that  $da$  does not depend on  $\theta$ , the averaged Ito equation of  $a(t)$  is independent of  $\theta(t)$  and that the random process  $a(t)$  is a 1-D diffusion process.

Thus, the reduced Fokker-Planck-Kolmogorov (FPK) equation of  $a(t)$  can be written:

$$0 = -\frac{\partial}{\partial a} [m_1(a)\rho(a)] + \frac{1}{2} \frac{\partial^2}{\partial a^2} [\sigma_{11}^2(a)\rho(a)] \tag{17}$$

The boundary conditions are:

$$\begin{aligned}
 \rho(a) &= c, \quad c \in (-\infty, +\infty) \quad \text{as } a = 0 \\
 \rho(a) &\rightarrow 0, \quad \frac{\partial \rho}{\partial a} \rightarrow 0 \quad \text{as } a \rightarrow \infty
 \end{aligned} \tag{18}$$

Based on the boundary conditions given in eq. (17), the amplitude stationary PDF can be obtained:

$$\rho(a) = \frac{C}{\sigma_{11}^2(a)} \exp \left[ \int_0^a \frac{2m_1(u)}{\sigma_{11}^2(u)} du \right] \tag{19}$$

where  $C$  is the normalized constant that satisfies:

$$C = \left\{ \int_0^\infty \left( \frac{1}{\sigma_{11}^2(a)} \exp \left[ \int_0^a \frac{2m_1(u)}{\sigma_{11}^2(u)} du \right] \right) da \right\}^{-1} \tag{20}$$

Substituting eq. (16) into eq. (19), we get the explicit expression of stationary PDF of the system amplitude:

$$\rho(a) = \frac{Cw_0^2(1+w_0^2\tau^2)}{\beta_1^2 D} a \exp \left( -\frac{\Delta}{1280\beta_1^2 D} \right) \tag{21}$$

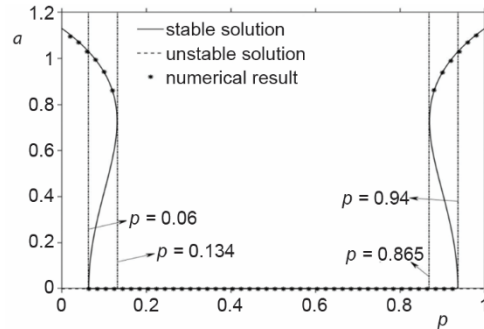
where  $C$  is the normalization constant and:

$$\Delta = \frac{1}{3} a^2 w_0^2 (1+w_0^2\tau^2) \left[ 1920w^p \cos\left(\frac{p\pi}{2}\right) + u_1(-240\alpha_1 a^2 + 80\alpha_2 a^4 + 1920\varepsilon) \right] \tag{22}$$

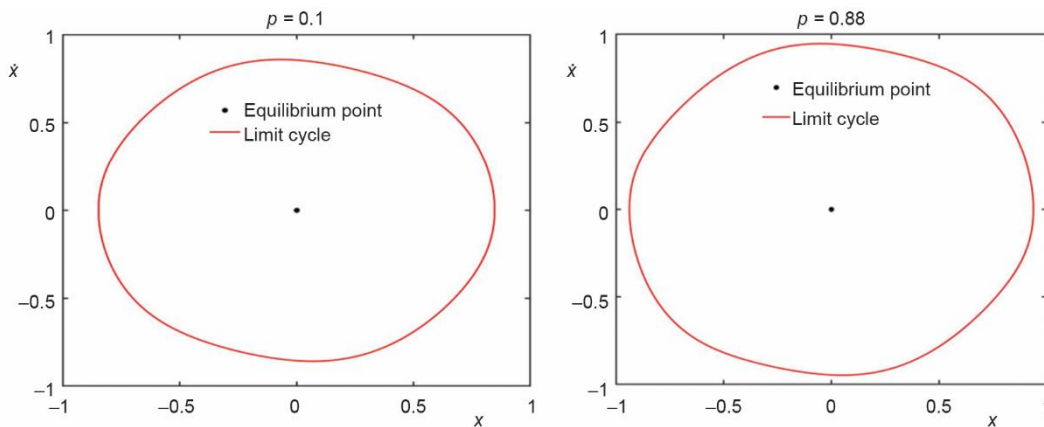
**Stochastic P-bifurcation of the system amplitude**

Firstly, we consider the case without the noise excitation in system (3), with the parameters  $\mu = 1$ ,  $\varepsilon = -0.1$ , the non-linear damping coefficients  $\alpha_1 = 1.51$ ,  $\alpha_2 = 2.85$ , and the natural frequency  $w = 1$ . For convenience in discussing parametric influence, the bifurcation diagram of the system amplitude with variation of the fractional derivative order  $p$  is shown in fig. 1 when  $D = 0$ .

As can be seen from fig. 1, there are only 1 attractor (large limit cycle) in the deterministic system when  $p$  changes in the intervals  $[0, 0.06]$  and  $[0.94, 1]$ , there is also only 1 attractor (equilibrium point) when  $p$  changes in the interval  $(0.134, 0.865)$ , and there are 2 attractors when  $p$  changes in the intervals  $(0.06, 0.134)$  and  $(0.865, 0.94)$ . And when  $p$  is taken as  $p = 0.1$  and  $p = 0.88$ , the phase diagrams are shown in the fig. 2, it also indicates that the deterministic system is bistable at this moment while  $p$  belongs to the intervals  $(0.06, 0.134)$  and  $(0.865, 0.94)$ .



**Figure 1. Bifurcation diagram of the deterministic system without noise excitation**



**Figure 2. Bistable phase diagram**

Stochastic P-bifurcation means that the changes in number of the stationary PDF curve peaks. To obtain the critical parametric conditions for stochastic P-bifurcation, we analyze the influences of parameters on the system stochastic P-bifurcation by using the singularity theory in this section.

For the sake of convenience,  $\rho(a)$  is expressed by:

$$\rho(a) = CR(a, D, \varepsilon, w, p, \alpha_1, \alpha_2) \exp[Q(a, D, \varepsilon, w, p, \alpha_1, \alpha_2)] \tag{23}$$

in which:

$$R(a, D, \varepsilon, w, p, \alpha_1, \alpha_2, u) = \frac{w_0^2(1+w_0^2\tau^2)}{\beta_1^2 D} a$$

$$Q(a, D, \varepsilon, w, p, \alpha_1, \alpha_2, u) =$$

$$= - \frac{a^2 w_0^2(1+w_0^2\tau^2) \left[ 1920w^p \cos\left(\frac{p\pi}{2}\right) + u_1(-240\alpha_1 a^2 + 80\alpha_2 a^4 + 1920\varepsilon) \right]}{3840\beta_1^2 D} \quad (24)$$

Based on the singularity theory [33], the stationary PDF of the system amplitude needs to satisfy:

$$\frac{\partial \rho(a)}{\partial a} = 0, \quad \frac{\partial^2 \rho(a)}{\partial a^2} = 0 \quad (25)$$

Substituting eq. (23) into eq. (25), we obtain:

$$H = \{R' + RQ' = 0, R'' + 2R'Q' + RQ'' + RQ'^2 = 0\} \quad (26)$$

where  $H$  is the condition for the changes in number of the PDF curve peaks.

### Taking $(p, D)$ as unfolding parameters

According to eqs. (24) and (26), we obtain the transition set for the system stochastic P-bifurcation with the unfolding parameters  $p$  and  $D$  shown in fig. 3.

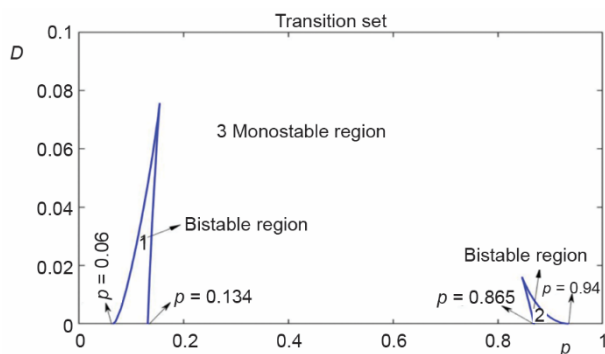
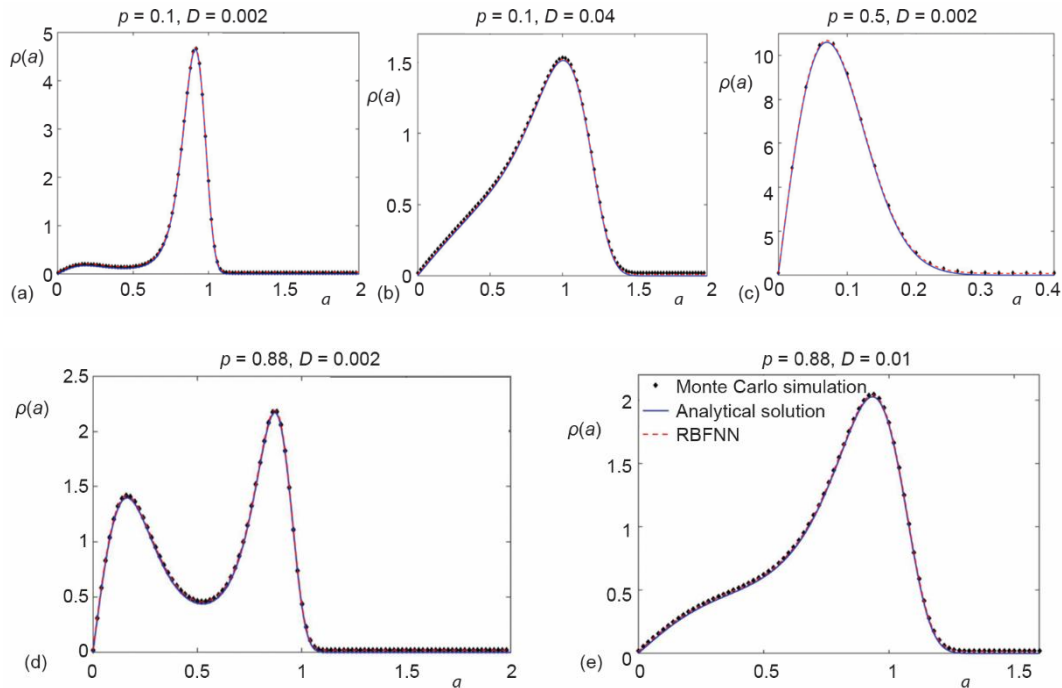


Figure 3. Transition set curves (taking  $p$  and  $D$  as the unfolding parameters)

Based on the singularity theory, we can obtain all varieties of the system stationary PDF curves that are qualitatively different. The unfolding parametric plane about  $p$  and  $D$  is divided into three sub-areas by the transition set curves. For the sake of convenience, each area in fig. 3 is marked with a number.

Without loss of generality, we analyze the stationary PDF of amplitude  $p(a)$  and the joint PDF  $p(x, \dot{x})$  only for one point  $(p, D)$  in each of the three sub-regions in fig. 3, and then compare the analytical solutions with the Monte Carlo simulation and radial basis function neural network (RBFNN) results from original system (3) using the numerical method for fractional derivative [20], and show the corresponding results in figs. 4 and 5, respectively.

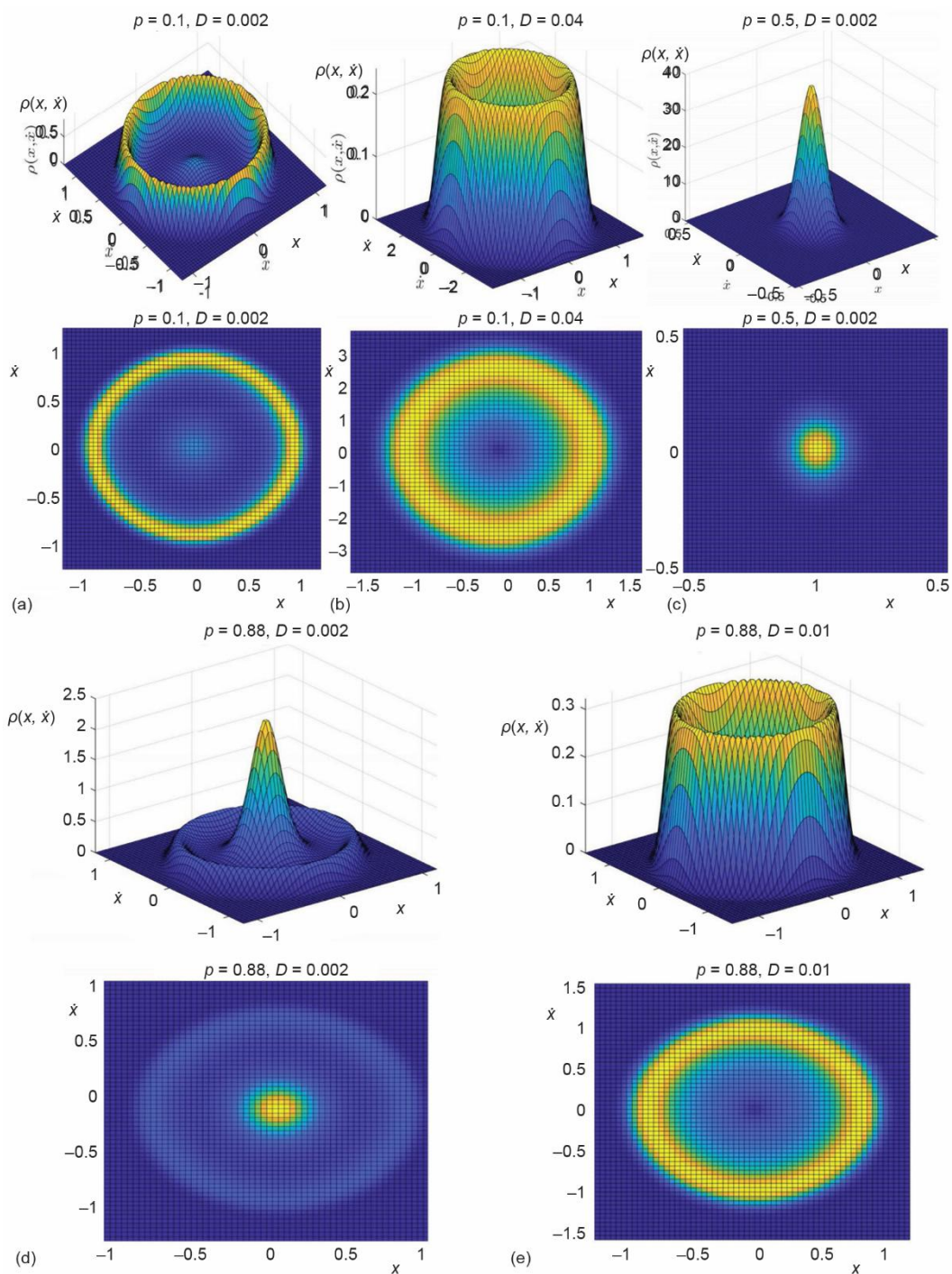


**Figure 4.** The PDF of  $p(a)$ , (taking  $p$  and  $D$  as the unfolding parameters) in each region of fig. 3; (a)  $(p, D)$  in region 1 of fig. 3, (b)  $(p, D)$  in region 3 of fig. 3, (c)  $(p, D)$  in region 3 of fig. 3, (d)  $(p, D)$  in region 2 of fig. 3, and (e)  $(p, D)$  in region 3 of fig 3

As shown in fig. 3, the unfolding parametric plane is segmented into three geometrically separated zones through the transition set curve. From fig. 3 we can see that, the parameter area where the PDF  $\rho(a)$  and  $\rho(x, \dot{x})$  occurs bimodal is surrounded by two curved triangle. When the parameter  $(p, D)$  is taken as  $p = 0.1, D = 0.002$  and  $p = 0.088, D = 0.002$  in area 1 and area 2, respectively, figs. 4(a), 5(a), 4(d), and 5(d), the PDF  $\rho(a)$  and  $\rho(x, \dot{x})$  has a stable limit cycle far away from the origin and the probability is not zero near the origin, there are both the limit cycle and equilibrium in the system simultaneously. When the parameter  $(p, D)$  is taken as other value in area 3, figs. 4(b), 5(b), 4(c), 5(c), 4(e), and 5(e), the PDF  $\rho(a)$  and  $\rho(x, \dot{x})$  has a remarkable peak near the original point (the system has an equilibrium point) or exhibits a prominent peak located at a significant distance from the origin (the system has a limit cycle), this means the system remains monostable in the area 3 at this moment.

Notwithstanding the precise values of the unfolding parameters, if they intersect any line in the transition set curve depicted in fig. 3, the system will undergo a stochastic P-bifurcation phenomenon. Therefore, the transition set curves are merely the critical parametric conditions of the system stochastic P-bifurcation. The analytic results displayed in fig. 4 are in close agreement with the numerical results obtained by Monte Carlo simulation and the RBFNN method from the original system (3), thereby further verifying the theoretical analysis and demonstrating the feasibility of employing the methods outlined in this paper to analyze the stochastic P-bifurcation behavior of fractional order systems.

In comparison with integral-order controllers, fractional-order controllers have been shown to exhibit superior dynamic performance and robustness [34]. In recent years, there has



**Figure 5.** Joint PDF and projection of (taking and as the unfolding parameters) in each region of fig. 3; (a)  $(p, D)$  in region 1 of fig. 3, (b)  $(p, D)$  in region 3 of fig. 3, (c)  $(p, D)$  in region 3 of fig. 3, (d)  $(p, D)$  in region 2 of fig. 3, and (e)  $(p, D)$  in region 3 of fig. 3

been significant progress in the development of fractional-order controllers [35-37]. In the preceding analysis, the regions in which the stochastic P-bifurcation occurs in system (3) were determined. These regions facilitate the system transition between monostable and bistable states through the corresponding unfolding parameters. This theoretical framework has the potential to provide a foundation for the analysis and design of fractional order controllers and can be easy to extend to fractal-fractional systems [38, 39].

### Conclusion

In this paper, we investigated the bistable phenomenon of a generalized Van der Pol system with the fractional inertial element driven by additive Gaussian colored noise excitation. We discussed the influences of the parameters, including the order of fractional derivative and the intensity of colored noise, on the system. We transformed the original system into an isovalent integer-order system by minimizing the mean square error. We acquired the stationary PDF of system amplitude via the stochastic averaging method. The results demonstrate that the order of the fractional derivative and the noise intensity can each induce stochastic P-bifurcation behavior in the system. Furthermore, the number of peaks in the stationary PDF curves of system amplitude can be regulated from one to two by selecting the corresponding unfolding parameters. By incorporating suitable unfolding parameters in accordance with the critical parametric conditions obtained, the system response can be constrained to diverse motion states, thereby demonstrating the intricate dynamic behaviors of the system with the fractional inertial force. These findings offer significant theoretical contributions to the field of system design in engineering disciplines such as mechanical and electrical engineering. The correlation between the numerical results obtained via the MCS and RBFNN methods and the analytical solutions serves to further validate our theoretical analysis. Furthermore, the findings of this study demonstrate the viability of the proposed methodology for the analysis of stochastic P-bifurcation behaviors of non-linear oscillators with fractional derivatives. Additionally, the methods and conclusions in [40, 41] can provide valuable references for improving the accuracy of frequency analysis in the studied system, particularly when dealing with complex non-linear factors and enhancing dynamic response prediction precision. Incorporating these findings into future research could lead to a deeper exploration of the system dynamic mechanisms and expand the scope of application of our theoretical framework in engineering practice.

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