LIE SYMMETRY CLASSIFICATION AND REDUCTION OF THE RIEMANN-LIOUVILLE FRACTIONAL RDCE WITH FOUR ARBITRARY FUNCTIONS

by

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The Lie symmetry classification of the fractional order reaction-diffusion-convection equation (RDCE) with four arbitrary functions in the sense of Riemann-Liouville fractional derivative is carried out by using the Lie symmetry analysis method. It is noteworthy that the equation retains two symmetries when it contains four arbitrary functions. When four arbitrary functions are substituted for concrete functions, the resulting equation exhibits enhanced symmetry. However, it is imperative to acknowledge that the Lie algebra space that encompasses fractional order RDCE is a sub-space of the Lie algebra of the integer order RDCE. In summary, it has been demonstrated that the initial equation is converted into a fractional ODE by the corresponding symmetry when four arbitrary functions are substituted with specific functions. This provides a foundation for further research, which may enhance our understanding of certain phenomena in life.

Key words: *fractional order RDCE, Riemann-Liouville fractional derivative, Lie symmetry, symmetry classifications*

Introduction

The study of fractional differential equations (FDE) emerged in the field of mathematics during the 20th century, emerging as a response to the limitations of traditional integer differential equations [1-3]. As computing power has advanced and the demand for understanding the behavior of complex systems has increased, the application of AI in science and engineering has garnered significant attention [4]. For instance, Rahioui *et al.* [5] employed FDE to model disease transmission in biology. Similarly, Ren and Zhang [6] utilized FDE to analyze medium diffusion phenomena in physics.

The Lie symmetry analysis method is regarded as one of the most general and effective methods for solving differential equations [7-9]. Its application to many fields, including wave motion, fluid mechanics, and astrophysics, among others, has been well-documented [5]. In the process of solving equations by employing Lie group theory, no assumptions must be made on the equations (or system of equations) [10]. Consequently, Lie group theory becomes a powerful tool for analyzing integer order differential equations [11-13]. Tian [14] carried out a Lie symmetry analysis of the heat conduction equation and the wave equation. Wang and Dong [15] used the Lie symmetry method to analyze the (2+1)-D generalized

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Painleve-Burgers system. Cherniha *et al.* [16] conducted an analysis of a class of reaction-diffusion-convection equations with integer order (1+2)-D by means of the Lie symmetry method. This analysis entailed the construction of its equivalent transformation group and the subsequent attainment of a complete Lie symmetry classification. In a separate study, Zuhal *et al.* [17] obtained the Lie point symmetric algebra of a novel coupled non-linear model. To enhance the construction efficiency of Lie algebra, Temur *et al.* [18] proposed an optimization algorithm based on Wu's method, which only needs to construct its isolated image without determining the Lie algebra. Tian and Wang [19] took a class of perturbation equations as an example and simplified the solution of overdetermined systems in the Lie symmetric method using polynomial characteristic methods. With the continuous improvement of Lie symmetry theory, this method has been used to analyze and solve many kinds of FDE. Gaur and Singh [20] derived group invariant solutions of fractional burgers-poisson equations by means of lie symmetry method. Dorjgotov *et al.* [21] carried out Lie symmetry analysis for a class of time-fractional non-linear evolutionary systems.

In recent years, considerable attention has been directed towards the Lie symmetry analysis of FDE with arbitrary functions. In their study, Gazizov *et al.* [22] examined the symmetry properties of fractional diffusion equations $\partial_t^\alpha u = [k(u)u_x]_x$, where ∂_t^α is the Riemann-Liouville or Caputo fractional derivative. Zhang [23] examined the fractional diffusion--convection equation $\partial_t^\alpha u = [P(u)u_x]_x + Q(u)u_x$, which Lie symmetry features are sorted out, where ∂_t^α represents the Riemann-Liouville fractional partial derivative. Zhang and Liu [24] considered Lie symmetry of the anomalous diffusion-type equation with variable-order fractional derivative $\partial_t^{\alpha(t)} u = [P(u)u_x]_x$. In addition, Hejazi *et al.* [25] identified the Lie point symmetry of the non-linear anomalous diffusion equation with the source term in anisotropic medium $\mathrm{D}_t^\alpha u = [f(u)u_x]_x + [g(u)u_y]_y + [h(u)u_z]_z + q(u)$ and so on.

In this paper, the fractional order RDCE [26] with four arbitrary functions is considered as follows:

$$\partial_t^\alpha u = [F(u)u_x]_x + [F(u)u_y]_y + K_1(u)u_x + K_2(u)u_y + R(u) \tag{1}$$

where $0 \le \alpha \le 1$, $u = (t, x)$, and $F(u), K_1(u), K_2(u), R(u)$ are all arbitrary functions, they are related to the three most common transport mechanisms that occur in real life. The diffusion coefficient $F(u) > 0$ is the main characteristic of the diffusion (thermal conductivity) process. The $K_1(u), K_2(u)$ usually represents velocity, which can be positive or negative, and can describe the convective transport. The reaction term $R(u)$ describes the dynamic process. This equation has been widely applied in physics, chemistry, engineering and so on.

Lie symmetry classification

Theorem 1. The Lie symmetry classification of fractional order RDCE of the form (1) that depend on $F(u)$, $K_1(u)$, $K_2(u)$, and $R(u)$ is given in tab. 1.

Proof 1. The invariant condition of eq. (1) can be obtained according to [27]:

$$\Big\{\eta^\alpha - \eta_1^{(2)}F(u) - \eta_2^{(2)}F(u) - \eta_1^{(1)}[F'(u)u_x + K_1(u)] - \eta_2^{(1)}[2F'(u)u_y + K_2(u)] -$$

$$-\eta[F'(u)u_{xx} + F'(u)u_{yy}] + F''(u)u_x^2 + F''(u)u_y^2 + K_1'(u)u_x + K_2'(u)u_y + R'(u)\Big\}\Big|_{(1)} = 0 \tag{2}$$

where

$$\begin{gathered}\eta^{\alpha}=\partial_t^{\alpha}\eta+[\eta_u-\alpha\mathrm{D}_t(\tau)]\partial_t^{\alpha}u-u\partial_t^{\alpha}(\eta_u)+\mu+\\ +\sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^{\alpha}(\eta_u)-\binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)\right]\partial_t^{\alpha-n}u-\\ -\binom{\alpha}{n}\mathrm{D}_t^{n}(\xi)\partial_t^{\alpha-n}(u_x)-\binom{\alpha}{n}\mathrm{D}_t^{n}(\zeta)\partial_t^{\alpha-n}(u_y)\end{gathered}\tag{3}$$

$$\mu=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}[-u(t,x,y)]^{r}}{\Gamma(n+1-\alpha)}\frac{\mathrm{d}\{[u(t,x,y)]^{k-r}\}}{\mathrm{d}t^{m}}\frac{\partial^{n-m+k}f(t,u)}{\partial t^{n-m}\partial u^{k}}\tag{4}$$

$$\eta_1^{(i)}=\mathrm{D}_x^{i}(\eta-\xi u_x-\tau u_t-\zeta u_y)+\xi u_{i+1}+\tau u_{it}+\zeta u_{iy}\tag{5}$$

$$\eta_2^{(j)}=\mathrm{D}_y^{j}(\eta-\xi u_x-\tau u_t-\zeta u_y)+\xi u_{jx}+\tau u_{jt}+\zeta u_{j+1}\tag{6}$$

Substituting $\eta^{\alpha}, \eta_1^{(i)}$, and $\eta_2^{(j)}$ into eq. (2) yields a determining system of equations:

$$\begin{gathered}\binom{\alpha}{n}\eta_t^{n}(\eta_u)-\binom{\alpha}{n+1}\mathrm{D}_t^{n+1}(\tau)=0, n=1,2,3,...\\ \xi_t=\xi_u=\tau_x=\tau_y=\tau_u=\zeta_t=\zeta_u=\eta_{uu}=0\\ 2F(u)(\xi_y+\zeta_x)=0\\ 2F'(u)(\xi_y+\zeta_x)=0\\ -\eta F'(u)-\alpha\tau_tF(u)+2\zeta_yF(u)=0\\ -\eta F'(u)-\alpha\tau_tF(u)+2\xi_xF(u)=0\\ -\eta K_1'(u)-\alpha\tau_tK_1(u)+\xi_yK_2(u)+\xi_{yy}F(u)+\xi_xK_1(u)+\\ +\xi_{xx}F(u)-2\eta_xF'(u)-2\eta_{xu}F(u)=0\\ -\eta K_2'(u)-\alpha\tau_tK_2(u)+\zeta_yK_2(u)+\zeta_{yy}F(u)+\zeta_xK_1(u)+\\ +\zeta_{xx}F(u)-2\eta_yF'(u)-2\eta_{yu}F(u)=0\\ -\eta F''(u)-(\alpha\tau_t-2\zeta_y+\eta_u)F'(u)-\eta_{uu}F(u)=0\\ -\alpha\tau_tF'(u)-\eta F''(u)+2\xi_xF'(u)-\eta_uF'(u)-\eta_{uu}F(u)=0\\ \partial_t^{\alpha}\eta-u\partial_t^{\alpha}(\eta_u)-\eta R'(u)-\alpha\tau_tR(u)+\eta_uR(u)-\eta_yK_2(u)-\eta_{yy}F(u)+\eta_xK_1(u)-\eta_{xx}F(u)=0\end{gathered}\tag{7}$$

Table 1. Infinitesimal operators of eq. (1)

$F(u)$	$K_1(u)$	$K_2(u)$	$R(u)$	Infinitesimal operators
u^k	0	0	0	$\partial_x, \partial_y, y\partial_x - x\partial_y$ $\frac{k}{2}x\partial_x + \frac{k}{2}y\partial_y + u\partial_u$ $\frac{[k(-1+\alpha)+2\alpha]x}{4}\partial_x + t\partial_t +$ $\frac{[k(-1+\alpha)+2\alpha]y}{4}\partial_y + \frac{[(-1+\alpha)]u}{4}\partial_u$
u^k	0	0	r_1u	$\partial_x, \partial_y, y\partial_x - x\partial_y$ $\frac{kx}{2}\partial_x + \frac{kx}{2}\partial_y + u\partial_u$
u^k	0	0	r_1u^γ $(\gamma \neq 1)$	$\partial_x, \partial_y, y\partial_x - x\partial_y$ $\frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)}x\partial_x + t\partial_t +$ $+\frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)}y\partial_y + \frac{\alpha}{1-\gamma}u\partial_u$
$u^{3-\gamma}$	k_1u	u	u^γ $(\gamma \neq 1)$	∂_x, ∂_y $\frac{\alpha(-2+\gamma)}{-1+\gamma}x\partial_x + t\partial_t +$ $+\frac{\alpha(-2+\gamma)}{-1+\gamma}y\partial_y + \frac{\alpha}{1-\gamma}u\partial_u$
u^k	u^μ	u^μ	$u^{1-k+2\mu}$ $(k \neq 2\mu)$	∂_x, ∂_y $\frac{\alpha(k-\mu)}{k-2\mu}x\partial_x + t\partial_t +$ $+\frac{\alpha(k-\mu)}{k-2\mu}y\partial_y + \frac{\alpha}{k-2\mu}u\partial_u$
u^k	$u^m \cos[p\ln x]$	$u^m \sin[p\ln x]$	σu^{2m+k-1} $(k \neq 2m)$	∂_x, ∂_y $\left[\frac{(k-m)\alpha}{k-2m}x - \frac{p\alpha}{k-2m}y\right]\partial_x + t\partial_t +$ $+\left[\frac{(k-m)\alpha}{k-2m}y + \frac{p\alpha}{k-2m}x\right]\partial_y +$ $+\frac{\alpha}{k-2m}u\partial_u$
u^k	$u^k\cos[p\ln x]$	$u^k\sin[p\ln x]$	σu^{k+1}	∂_x, ∂_y $\frac{p\alpha}{k}y\partial_x + t\partial_t - \frac{p\alpha}{k}x\partial_y - \frac{\alpha}{k}u\partial_u$
Arbitrary function	Arbitrary function	Arbitrary function	Arbitrary function	∂_x, ∂_y

According to the second equation in the system of eq. (7), it can be inferred that $\xi=\xi(x,y),\ \tau=\tau(t),\ \zeta=\zeta(x,y),\ \eta=\eta_1(t,x,y)u+\eta_2(t,x,y)$. Assuming $\tau(t)=b_2t^2+b_1t$ according to [23]. Observing the previous system of equations, it can also be seen that $\xi_y=-\zeta_x,\xi_x=\zeta_y$.

- Equation (1) has the following symmetries, when $F(u)=u^k, K_1(u)=0,\ K_2(u)=0,\ R(u)=0$, where k is any non-zero constant:

$$\begin{gathered}\xi=\frac{1}{4}x\{2a_1k+b_1[k(-1+\alpha)+2\alpha]\}+c_{11}y+c_{12},\quad \tau=b_1t\\ \zeta=\frac{1}{4}y\{2a_1k+b_1[k(-1+\alpha)+2\alpha]\}-c_{11}x+c_{13},\quad \eta=\frac{1}{2}u[2a_1+b_1(-1+\alpha)]\end{gathered}\tag{8}$$

- When $F(u)=u^k, K_1(u)=0,\ K_2(u)=0,\ R(u)=r_1u$, where k, r_1 is any non-zero constant, the symmetries possessed by eq. (1) can be obtained:

$$\xi=\frac{a_1k}{2}x+c_{11}y+c_{12},\quad \tau=0,\quad \zeta=\frac{a_1k}{2}y-c_{11}x+c_{13},\quad \eta=a_1u\tag{9}$$

- When $F(u)=u^k, K_1(u)=0,\ K_2(u)=0,\ R(u)=r_1u^\gamma$, where k, r_1, γ is any non-zero constant, the symmetries of eq. (1) can be denoted:

$$\begin{gathered}\xi=\frac{b_1\alpha(-1-k-\gamma)}{2(-1+\gamma)}x+c_{11}y+c_{12},\quad \tau=b_1t\\ \zeta=\frac{b_1\alpha(-1-k-\gamma)}{2(-1+\gamma)}y-c_{11}x+c_{13},\quad \eta=\frac{b_1\alpha}{1-\gamma}u\end{gathered}\tag{10}$$

- Equation (1) has the following symmetries, when $F(u)=u^{3-\gamma}, K_1(u)=k_1u,\ K_2(u)=u,\ R(u)=u^\gamma$, where $\gamma\neq 1$, k is any non-zero constant:

$$\xi=\frac{b_1\alpha(-2+\gamma)}{-1+\gamma}x+c_{12},\quad \tau=b_1t,\quad \zeta=\frac{b_1\alpha(-2+\gamma)}{-1+\gamma}y+c_{13},\quad \eta=\frac{b_1\alpha u}{1-\gamma}\tag{11}$$

- When $F(u)=u^k,\ K_1(u)=u^\lambda,\ K_2(u)=u^\mu,\ R(u)=u^\gamma$, where k, λ, μ, γ is any non-zero constant, and $k\neq 2\mu,$ the symmetries possessed by eq. (1) can be obtained:

$$\xi=\frac{b_1\alpha(k-\mu)}{k-2\mu}x+c_{12},\quad \tau=b_1t,\quad \zeta=\frac{b_1\alpha(k-\mu)}{k-2\mu}y+c_{13},\quad \eta=\frac{b_1\alpha u}{k-2\mu}\tag{12}$$

- When $F(u)=u^k,\ K_1(u)=u^m\cos[p\ln x],\ K_2(u)=u^m\sin[p\ln x],\ R(u)=\sigma u^{2m+k-1}$, where k,p,m,σ is any non-zero constant, and $k\neq 2m$, the symmetries possessed by eq. (1) can be denoted:

$$\begin{gathered}\xi=\frac{b_1\alpha(-1-k-\gamma)}{2(-1+\gamma)}x+c_{11}y+c_{12},\quad \tau=b_1t\\ \zeta=\frac{b_1\alpha(-1-k-\gamma)}{2(-1+\gamma)}y-c_{11}x+c_{13},\quad \eta=\frac{b_1\alpha}{1-\gamma}u\end{gathered}\tag{13}$$

- Equation (1) has the following symmetries, when $F(u)=u^k, K_1(u)=u^k\cos[p\ln x],\ K_2(u)=u^k\sin[p\ln x],\ R(u)=\sigma u^{k+1}$, where k,p,σ is any non-zero constant:

$$\xi = \frac{b_1 p\alpha}{k} y + c_{12}, \quad \tau = b_1 t, \quad \zeta = -\frac{b_1 p\alpha}{k} x + c_{13}, \quad \eta = -\frac{b_1 \alpha u}{k} \tag{14}$$

– Equation (1) has the following symmetries, when $F(u), K_1(u), K_2(u)$, and $R(u)$ all take any non-zero constant:

$$\xi = c_1, \quad \tau = 0, \quad \zeta = c_2, \quad \eta = 0 \tag{15}$$

From the previosu conclusions, it is easy to get the corresponding infinitesimal as shown in tab. 1.

Lie symmetry classification

Theorem 2. When $F(u) = u^{3-\gamma}$, $K_1(u) = k_1 u,\ K_2(u) = u,\ R(u) = u^{\gamma} (\gamma \neq 1)$, in the case of infinitesimal operator:

$$X_3 = \frac{\alpha(-2+\gamma)}{-1+\gamma} x\partial_x + t\partial_t + \frac{\alpha(-2+\gamma)}{-1+\gamma} y\partial_y + \frac{\alpha}{1-\gamma} u\partial_u$$

eq. (1) can be reduced to a fractional order differential equation:

$$\left[P_{\frac{-1+\gamma}{\alpha(-2+\gamma)}}^{1+\frac{\gamma\alpha}{1-\gamma},\alpha} S \right](\omega) = (3-\gamma)[S(\omega)]^{2-\gamma}[S'(\omega)]^2 + [S(\omega)]^{3-\gamma} S''(\omega) + k_1 S(\omega) S'(\omega) + [S(\omega)]^{\gamma} \tag{16}$$

where

$$\omega = xt^{-\frac{\alpha(-2+\gamma)}{-1+\gamma}}, \quad S'(\omega) = \frac{\mathrm{d}S}{\mathrm{d}\omega}, \text{ and } (P_{\delta}^{\nu,\kappa})$$

is the Erdélyi-Kober fractional order differential operator.

Proof 2. Consider the infinitesimal operator:

$$X_3 = \frac{\alpha(-2+\gamma)}{-1+\gamma} x\partial_x + t\partial_t + \frac{\alpha(-2+\gamma)}{-1+\gamma} y\partial_y + \frac{\alpha}{1-\gamma} u\partial_u$$

and solve its characteristic equation:

$$\frac{-1+\gamma}{\alpha(-2+\gamma)} \frac{\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}t}{t} = \frac{1-\gamma}{\alpha} \frac{\mathrm{d}u}{u} \tag{17}$$

Similar variables:

$$u = S(\omega) t^{\frac{\alpha}{1-\gamma}} \text{ and } \omega = xt^{-\frac{\alpha(-2+\gamma)}{-1+\gamma}}$$

can be obtained.

Assuming $\sigma = t/r$, and the α order derivative of u can be expressed in the following form:

$$\partial_t^{\alpha} u = \partial_t^{\alpha} [S(\omega) t^{\frac{\alpha}{1-\gamma}}] = \frac{\partial}{\partial t} \left\{ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} r^{\frac{\alpha}{1-\gamma}} S\left[xr^{-\frac{\alpha(-2+\gamma)}{-1+\gamma}} \right] \mathrm{d}r \right\} =$$

$$= \frac{\partial}{\partial t} \left\{ \left(t^{1+\frac{\gamma\alpha}{1-\gamma}} \right) \left[K_{\frac{-1+\gamma}{\alpha(-2+\gamma)}}^{1+\frac{\alpha}{1-\gamma},1-\alpha} S \right](\omega) \right\} = t^{\frac{\gamma\alpha}{1-\gamma}} \left[P_{\frac{-1+\gamma}{\alpha(-2+\gamma)}}^{1+\frac{\alpha}{1-\gamma},\alpha} S \right](\omega) \tag{18}$$

So:

$$u_x = t^{\frac{\alpha(\gamma-1)}{1-\gamma}} S'(\omega), \quad u_{xx} = t^{\frac{\alpha(2\gamma-3)}{1-\gamma}} S''(\omega)$$

By substituting the previous results into eq. (1), the reduced equation can be obtained:

$$\left[P_{\frac{-1+\gamma}{\alpha(-2+\gamma)}}^{1+\frac{\gamma\alpha}{1-\gamma},\alpha} S \right](\omega) = (3-\gamma)[S(\omega)]^{2-\gamma}[S'(\omega)]^2 + [S(\omega)]^{3-\gamma} S''(\omega) + k_1 S(\omega) S'(\omega) + [S(\omega)]^{\gamma} \tag{19}$$

Theorem 3. When $F(u) = u^k, K_1(u) = 0,\ K_2(u) = 0,\ R(u) = r_1 u^{\gamma}$, in the case of infinitesimal operator:

$$X_4 = \frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)} x\partial_x + t\partial_t + \frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)} y\partial_y + \frac{\alpha}{1-\gamma} u\partial_u$$

eq. (1) can be reduced to a fractional order differential equation:

$$\left[P_{\frac{2(-1+\gamma)}{\alpha(-1-k+\gamma)}}^{1+\frac{\gamma\alpha}{1-\gamma},\alpha} S \right](\omega) = k[S(\omega)]^{k-1}[S'(\omega)]^2 + [S(\omega)]^k S''(\omega) + r_1 [S(\omega)]^{\gamma} \tag{20}$$

where

$$\omega = x t^{-\frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)}}, \quad S'(\omega) = \frac{\mathrm{d}S}{\mathrm{d}\omega}$$

Proof 3. Consider the infinitesimal operator:

$$X_4 = \frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)} x\partial_x + t\partial_t + \frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)} y\partial_y + \frac{\alpha}{1-\gamma} u\partial_u$$

and solve its characteristic equation:

$$\frac{2(-1+\gamma)}{\alpha(-1-k+\gamma)} \frac{\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}t}{t} = \frac{1-\gamma}{\alpha} \frac{\mathrm{d}u}{u} \tag{21}$$

Similar variables:

$$u = S(\omega) t^{\frac{\alpha}{1-\gamma}} \quad \text{and} \quad \omega = x t^{-\frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)}}$$

can be obtained.

Assuming $\sigma = t/r$, and the α order derivative of u can be expressed in the following form:

$$\partial_t^{\alpha} u = \partial_t^{\alpha} [S(\omega) t^{\frac{\alpha}{1-\gamma}}] = \frac{\partial}{\partial t} \left[\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-r)^{-\alpha} r^{\frac{\alpha}{1-\gamma}} S\left[x r^{-\frac{\alpha(-1-k+\gamma)}{2(-1+\gamma)}} \right] \mathrm{d}r \right] =$$

$$= \frac{\partial}{\partial t} \left[\left(t^{1+\frac{\gamma\alpha}{1-\gamma}} \right) \left[K_{\frac{2(-1+\gamma)}{\alpha(-1-k+\gamma)}}^{1+\frac{\alpha}{1-\gamma},1-\alpha} S \right](\omega) \right] = t^{\frac{\gamma\alpha}{1-\gamma}} \left[P_{\frac{2(-1+\gamma)}{\alpha(-1-k+\gamma)}}^{1+\frac{\alpha}{1-\gamma},\alpha} S \right](\omega) \tag{22}$$

So:

$$u_x = t^{\frac{\alpha(1-k+\gamma)}{2(1-\gamma)}} S'(\omega), \quad u_{xx} = t^{\frac{\alpha(-k+\gamma)}{1-\gamma}} S''(\omega)$$

By substituting the previous results into eq. (1), the reduced equation can be obtained:

$$\left[P^{1+\frac{\gamma\alpha}{1-\gamma},\alpha}_{\frac{2(-1+\gamma)}{\alpha(-1-k+\gamma)}} S\right](\omega) = k[S(\omega)]^{k-1}[S'(\omega)]^2 + [S(\omega)]^k S''(\omega) + r_1[S(\omega)]^{\gamma} \tag{23}$$

Conclusion

In this paper, the Lie symmetry classification of the fractional order RDCE with Riemann-Liouville fractional derivatives is presented. It is noteworthy that the equation retains two symmetries when it contains four arbitrary functions. When four arbitrary functions are substituted for concrete functions, the resulting equation exhibits enhanced symmetry. However, it is important to note that the Lie algebra space that admits fractional order RDCE is a sub-space of the Lie algebra of the integer order RDCE. In conclusion, the reduction of the original equation was achieved through the application of the corresponding symmetry when four arbitrary functions were converted into concrete functions. The findings of this study provide a solid foundation for future research on the equation, particularly in terms of its further resolution.

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References

[1] Feng, G. Q.. A Circular Sector Vibration System in a Porous Medium: A Fractal-Fractional Model and He's Frequency Formulation, *Facta Universitatis Series: Mechanical Engineering*, *23* (2025), 2, pp. 377-385

[2] Hendy, M. H., *et al.*, A Problem in Fractional Order Thermo-Viscoelasticity Theory for a Polymer Micro-Rod with and without Energy Dissipation, *Advances in Differential Equations and Control Processes*, *31* (2024), 4, pp. 583-607

[3] He, C.-H., *et al.*, A Fractal-Based Approach to the Mechanical Properties of Recycled Aggregate Concretes, *Facta Universitatis, Series: Mechanical Engineering, 22* (2024), 2, pp. 329-342

[4] Lei, X. H., He, J.-H., Frontiers in Thermal Science Driven by Artificial Intelligence, *Thermal Science, 29* (2025), 3A, pp. 1671-1677

[5] Rahioui, M., *et al.*, Lie Symmetry Analysis and Conservation Laws for the Time Fractional Generalized Advection-Diffusion Equation, *Computational and Applied Mathematics*, *42* (2023), 50

[6] Ren, R. C., Zhang, S. L., Invariant Analysis, Conservation Laws, and Some Exact Solutions for (2+1)-Dimension Fractional Long-Wave Dispersive System, *Computational and Applied Mathematics, 39* (2020), 4, pp. 4125-4135

[7] Sun, H., *et al.*, A New Collection of Real World Applications of Fractional Calculus in Science and Engineering, *Communications in Nonlinear Science and Numerical Simulation, 64* (2018), Nov., pp. 213-231

[8] Guo, B. L., *Fractional Partial Differential Equations and Their Numerical Solutions,* Science Press, Beijing, China, 2015

[9] Wang, G. W., Xu, T. Z., Invariant Analysis and Exact Solutions of Nonlinear Time Fractional Sharma--Tasso-Olver Equation by Lie Group Analysis, *Nonlinear Dynamics, 76* (2014), 1, pp. 571-580

[10] San, S., Yasar, E., On the Lie Symmetry Analysis, Analytic Series Solutions, and Conservation Laws of the Time Fractional Belousov-Zhabotinskii System, *Nonlinear Dynamics*, *109* (2022), 4, pp. 2997-3008
[11] Olver, P. J., *Applications of Lie Groups to Differential Equations,* Springer, New York, USA, 1993
[12] Bluman, G. W., *et al.*, *Applications of Symmetry Methods to Partial Differential Equations,* Springer New York, USA, 2010
[13] Ovsiannikov, L. V., *Group Analysis of Differential Equations,* Academic Press, New York, USA, 1982
[14] Tian, Y., Symmetry Reduction a Promising Method for Heat Conduction Equations, *Thermal Science, 23* (2019), 4, pp. 2219-2227
[15] Wang, Y., Dong, Z., Symmetry Analysis of a (2+1)-D System, *Thermal Science, 22* (2018), 4, pp. 1811-1822
[16] Cherniha, R., *et al*., A Complete Lie Symmetry Classification of a Class of (1+2)-Dimensional Reaction-Diffusion-Convection Equations, *Communications in Nonlinear Science and Numerical Simulation, 92* (2021), 105466
[17] Zuhal, K., *et al.*, On Exact Solutions for New Coupled Nonlinear Models Getting Evolution of Curves in Galilean Space, *Thermal Science*, *23* (2019), Suppl. 1, pp. S227-S233
[18] Temuer, C. L., *et al.*, A Mechanized Algorithm for Determining the Structural Constants of Symmetric Lie Algebras of Differential Equations Based on Wu's Method (in Chinese), *Scientia Sinica (Mathematica), 49* (2019), 5, pp. 751-764
[19] Tian, Y., Wang, L. K., Polynomial Characteristic Method an Easy Approach to Lie Symmetry, *Thermal Science, 24* (2020), 4, pp. 2629-2635
[20] Gaur, M., Singh, K., On Group Invariant Solutions of Fractional Order Burgers-Poisson Equation, *Applied Mathematics and Computation, 244* (2014), Oct., pp. 870-877
[21] Dorjgotov, K., *et al.*, Lie Symmetry Analysis of a Class of Time Fractional Nonlinear Evolution Systems, *Applied Mathematics and Computation, 329* (2018), July, pp. 105-117
[22] Gazizov, K. R., *et al.*, Symmetry Properties of Fractional Diffusion Equations, *Physica Scripta, 136* (2009), 5
[23] Zhang, Z. Y., Symmetry Determination and Nonlinearization of a Nonlinear Time-Fractional Partial Differential Equation, *Proceedings of the royal society A-Mathematical Physical and Engineering Sciences, 476* (2020), 2233, 20190564
[24] Zhang, Z. Y., Liu C. B., Leibniz-Type Rule of Variable-Order Fractional Derivative and Application to Build Lie Symmetry Framework, *Applied Mathematics and Computation*, *430* (2022), 127268
[25] Hejazi, S. R., *et al.*, Anisotropic Nonlinear Time-Fractional Diffusion Equation with a Source Term: Classification via Lie Point Symmetries, Analytic Solutions and Numerical Simulation, *Applied Mathematics and Computation*, *391* (2021), 125652
[26] Prakash, P., *et al.*, Initial Value Problem for the (2+1)-Dimensional Time-Fractional Generalized Convection-Reaction-Diffusion Wave Equation: Invariant Subspaces and Exact Solutions, *Computational and Applied Mathematics*, *41* (2022), 30
[27] Zhang, Z. Y., Zheng, J., Symmetry Structure of Multi-Dimensional Time-Fractional Partial Differential Equations, *Nonlinearity*, *34* (2021), 8, pp. 5186-5212

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