OSCILLATION OF A CLASS OF NON-LINEAR NEUTRAL TYPE DAMPED DIFFERENTIAL EQUATIONS

by

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Original scientific paper
https://doi.org/10.2298/TSCI2602125L

The objective of this paper is to present novel sufficient conditions for the oscillation of all solutions of a class of second-order neutral type damped non-linear differential equations by using the Riccati transformation and integral inequality technique. The merits of the obtained results are illustrated by the following example.

Key words: *differential equation, oscillation criteria, Riccati transformation*

Introduction

Oscillation phenomena are present in many models derived from real-world applications [1-3], such as non-Newtonian fluid theory and turbulent flow of polytrophic gases in porous media [4-6]. Became a hot topic recently in both mathematics and engineering, because of the importance of the oscillation of the second-order differential equations.

We consider an important second-order non-linear oscillation equation which involves damping term, which plays an important role in physics and engineering:

$$(r(t)\phi_\alpha\{z'(t)]\} + p(t)\phi_\alpha[z'(t)] + \int_a^b q(t,\xi) f\{x[\sigma(t,\xi)]\}\mathrm{d}\xi = 0, \quad t \ge t_0 \tag{1}$$

where

$$z(t) = x(t) + \int_c^d g(t,\xi)x[\tau(t,\xi)]\mathrm{d}\xi, \quad f\{x[\sigma(t,\xi)]\} \ge kx^\beta[\sigma(t,\xi)], \quad \phi_\lambda(s) = |s|^{\lambda-1}s$$

$$0 < a < b, \quad 0 < c < d, \quad \alpha > 0, \quad \beta > 0, \quad k > 0$$

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We assume that the following conditions hold:

$$(h_1)\ r\in C^1([t_0,\infty),\ R_+)\,,\ r'(t)\geq 0$$

$$(h_2)\ \tau\in C^2([t_0,\infty)\times[c,d],\ \ R_+)\,,\tau(t,\xi)\leq t\ \ ,\lim_{t\to\infty}\tau(t,\xi)=\infty$$

$$(h_3)\ \sigma\in C^1([t_0,\infty)\times[a,b],\ R_+)\,,\ \sigma(t,\xi)$$

is a decreasing function with respect to ξ:

$$\sigma(t,\xi)\leq t\,,\lim_{t\to\infty}\sigma(t,\xi)=\infty\,,\ \frac{d\sigma(t,\xi)}{dt}\geq 0$$

$$(h_4)\ q\in C([t_0,\infty)\times[a,b],\ R_+)\,,\,g\in C([t_0,\infty)\times[c,d],\ R_+)$$

Many results have been obtained from the oscillation study of second-order differential equation. In [7], a second-order non-linear oscillation system is studied, and the important mechanism of periodic motion is revealed by using the amplitude-frequency formula of non-linear oscillator. Some results have been obtained for the special case of $\alpha = \beta = 1$ without damping term under regular conditions [8, 9]. Some results have been obtained for the special case of $\alpha = \beta > 0$ without damping term under regular conditions [10-14]. Some results have been obtained for the special case of $\alpha = \beta > 0$ without damping term under non-regular conditions [15, 16]. Some results have been obtained with damping term under regular conditions [17, 18]. Some results have been obtained with damping term under non-regular conditions [19]. Some results have been obtained without damping term under regular conditions [20]. Some results have been obtained without damping term under non-regular or regular conditions [21-23].

In this paper, we study eq. (1) in the cases $\alpha > \beta$ or $\alpha \leq \beta$. Oscillation criteria for eq. (1) have been derived. The advantages of the results obtained are illustrated by an example.

Oscillation criteria

In this paper, we study the regular form:

$$\Phi(t)=\int\limits_t^\infty R^{-1/\alpha}(s)\mathrm{d}s=\infty,\quad t\geq t_0 \tag{2}$$

where $R(t)=E(t)r(t)$, $E(t)=\exp\int\limits_{t_0}^t\dfrac{p(s)}{r(s)}\,\mathrm{d}s$.

Lemma 1. If $x(t)$ is an eventually positive solution of eq. (1) and hypothesis (2) holds, then $z'(t) > 0$ and have the following formula:

$$\{R(t)[z'(t)]^\alpha\}'+A(t)z^\beta[\sigma(t,b)]\leq 0,\quad t\geq t_0 \tag{3}$$

where

$$A(t)=kE(t)\int\limits_a^b q(t,\xi)\left\{1-\int\limits_c^d g[\sigma(t,\xi],u)\mathrm{d}u\right\}^\beta \mathrm{d}\xi,\quad k>0$$

Proof. Equation (1) is multiplied by $E(t)$, we get:

$$\{R(t)\phi_\alpha[z'(t)]\}' + E(t)\int_a^b q(t,\xi) f\{x[\sigma(t,\xi)]\}\mathrm{d}\xi = 0, \quad t \ge t_0 \tag{4}$$

Since $x(t)$ is an eventually positive solution to eq. (1), there exists a $t_1 \ge t_0$ such that $x[\sigma(t,\xi)] > 0,\ \ x[\tau(t,\xi)] > 0$ for $t \ge t_1$, we have:

$$\{R(t)\phi_\alpha[z'(t)]\}' = [R(t)|z'(t)|^{\alpha\text{-}1} z'(t)]' \le 0, \quad t \ge t_1$$

hence $R(t)|z'(t)|^{\alpha\text{-}1} z'(t)$ is a decreasing function for $t \ge t_1$, therefore $z'(t)$ is also of one sign. Thus, there exists a $t_2 \ge t_1$ such that $z'(t) > 0$ or $z'(t) < 0$ for $t \ge t_1$.

Suppose that $z'(t) < 0$ for $t \ge t_2$, since $R(t)|z'(t)|^{\alpha\text{-}1} z'(t)$ is a decreasing function for $t \ge t_2$, there exists a $T_1 \ge t_2$, such that:

$$R(t)|z'(t)|^{\alpha\text{-}1} z'(t) = -R(t)[-z'(t)]^\alpha \le -R(T_1)[-z'(T_1)]^\alpha, \quad t \ge T_1,$$

let $k_1 = R(T_1)[-z'(T_1)]^\alpha > 0,$ we get:

$$R(t)|z'(t)|^{\alpha\text{-}1} z'(t) \le -k_1 < 0,\ t \ge T_1$$

That implies:

$$R(t)[-z'(t)]^\alpha \ge k_1, \quad t \ge T_1$$

That is:

$$-z'(t) \ge k_1^{1/\alpha} R^{-1/\alpha}(t), \quad t \ge T_1$$

We, therefore, obtain:

$$0 \le z(t) \le z(T_1) - k_1^{1/\alpha}\int_{T_1}^t R^{-1/\alpha}(s)\,\mathrm{d}s, \quad t \ge T_1$$

From eq. (2), we can know the contradiction. Therefore, we get $z'(t) > 0$ for $t \ge t_2$. Defined by $z(t)$, we have:

$$x(t) = z(t) - \int_c^d g(t,\xi) x[\tau(t,\xi)]\mathrm{d}\xi \le z(t), \quad t \ge t_1$$

From $z'(t) > 0$ for $t \ge t_2$, $z(t)$ is a increasing function for $t \ge t_2$. By condition (h_2), $\tau(t,\xi) \le t$, $\lim_{t\to\infty}\tau(t,\xi) = \infty$, there exists a $t_3 \ge t_2$, such that $\tau(t,\xi) \ge t_2$ for $t \ge t_3$, we obtain $z(\tau(t,\xi)) \le z(t)$, $t \ge t_3$, and:

$$x(t) = z(t) - \int_c^d g(t,\xi)x[\tau(t,\xi)]\mathrm{d}\xi \ge z(t) - \int_c^d g(t,\xi) z[\tau(t,\xi)]\mathrm{d}\xi \ge z(t)\left\{1 - \int_c^d g(t,\xi)\mathrm{d}\xi\right\}, \quad t \ge t_3$$

By condition (h_3), $\lim\sigma(t,\xi) = \infty$, there exists a $t_4 \ge t_3$, such that $\sigma(t,\xi) \ge t_3$ for $t \ge t_4$:

$$x[\sigma(t,\xi)] \ge z[\sigma(t,\xi)]\left\{1 - \int_c^d g[\sigma(t,\xi),u]\mathrm{d}u\right\}, \quad t \ge t_4$$

where $\sigma(t,\xi)$ is a decreasing function with respect to ξ, we get:

$$\sigma(t,\xi) \ge \sigma(t,b), \quad a \le \xi \le b, \quad t \ge t_0$$

Considering $z(t)$ is an increasing function for $t \geq t_2$, we have:

$$z[\sigma(t,\xi)] \geq z[\sigma(t,b)], \quad t \geq t_4$$

and

$$\int_a^b q(t,\xi) f\{x[\sigma(t,\xi)]\}\mathrm{d}\xi \geq k\int_a^b q(t,\xi) x^\beta[\sigma(t,\xi)]\mathrm{d}\xi \geq$$

$$\geq k\int_a^b q(t,\xi) z^\beta[\sigma(t,\xi)]\left\{1-\int_c^d g[\sigma(t,\xi),u]\mathrm{d}u\right\}^\beta \mathrm{d}\xi \geq$$

$$\geq kz^\beta[\sigma(t,b)]\int_a^b q(t,\xi)\left\{1-\int_c^d g[\sigma(t,\xi),u]\mathrm{d}u\right\}^\beta \mathrm{d}\xi, \quad t \geq t_4$$

From eq. (4), we have:

$$\{R(t)[z'(t)]^\alpha\}' + kE(t)z^\beta[\sigma(t,b)]\int_a^b q(t,\xi)\left\{1-\int_c^d g[\sigma(t,\xi),u]\,\mathrm{d}u\right\}^\beta \mathrm{d}\xi \leq 0, \quad t \geq t_4$$

Letting:

$$A(t) = kE(t)\int_a^b q(t,\xi)\left\{1-\int_c^d g[\sigma(t,\xi),u]\,\mathrm{d}u\right\}^\beta \mathrm{d}\xi$$

we get:

$$\{R(t)[z'(t)]^\alpha\}' + A(t)z^\beta[\sigma(t,b)] \leq 0, \quad t \geq t_4$$

Lemma 2. Hypothesis (2) holds, then:

$$V'(t) + A(t) + B(t)V^{(\lambda+1)/\lambda}(t) \leq 0 \tag{5}$$

where

$$V(t) = \frac{R(t)[z'(t)]^\alpha}{z^\beta[\sigma(t,b)]}, \quad A(t) = kE(t)\int_a^b q(t,\xi)\left\{1-\int_c^d g[\sigma(t,\xi),u]\,\mathrm{d}u\right\}^\beta \mathrm{d}\xi$$

$$\lambda = \min\{\alpha,\beta\}, \quad k_2 = \min\left\{1,\ [z'(T_2)]^{\frac{\beta-\alpha}{\beta}}\right\}, \quad k_3 = \min\left(1,\ \left\{z[\sigma(T_3,b)]^{\frac{\beta-\alpha}{\alpha}}\right\}\right)$$

$$k_4 = \min\{k_2,\ k_3\}, \quad B(t) = \frac{\lambda k_4\sigma'(t,b)}{R^{\frac{1}{\lambda}}[\sigma(t,b)]}$$

Proof. (I) If $\alpha > \beta$, we get:

$$V'(t)=\frac{\{R(t)[z'(t)]^{\alpha}\}'z^{\beta}[\sigma(t,b)]-R(t)[z'(t)]^{\alpha}\beta z^{\beta-1}[\sigma(t,b)]z'[\sigma(t,b)]\sigma'(t,b)}{z^{2\beta}[\sigma(t,b)]}=$$

$$=\frac{\{R(t)[z'(t)]^{\alpha}\}'}{z^{\beta}[\sigma(t,b)]}-\frac{\beta\sigma'(t,b)R(t)[z'(t)]^{\alpha}z'[\sigma(t,b)]}{z^{\beta+1}[\sigma(t,b)]}$$

From *Lemma 1*, we get:

$$V'(t)\le -A(t)-\frac{\beta\sigma'(t,b)}{R^{1/\beta}(t)}z'[\sigma(t,b)](z'(t))^{-\alpha/\beta}\left\{\frac{R(t)[z'(t)]^{\alpha}}{z^{\beta}[\sigma(t,b)]}\right\}^{(\beta+1)/\beta}=$$

$$=-A(t)-\frac{\beta\sigma'(t,b)}{R^{1/\beta}(t)}z'[\sigma(t,b)][z'(t)]^{-\alpha/\beta}V^{(\beta+1)/\beta}(t),\quad t\ge t_4$$

$$\{R(t)[z'(t)]^{\alpha}\}'=R'(t)[z'(t)]^{\alpha}+\alpha R(t)[z'(t)]^{\alpha-1}z''(t)\le 0,\quad t\ge t_4$$

Since $R(t) > 0$ and $R'(t) = E(t)p(t) + E(t)r'(t) > 0$ for $t \ge t_0$, $z'(t) > 0$ for $t \ge t_2$, then $z''(t) < 0$ for $t \ge t_4$. The $z'(t)$ is a decreasing function for $t \ge t_4$. Since:

$$\sigma(t,\xi)\le t,\quad \lim_{t\to\infty}\sigma(t,\xi)=\infty$$

there exists a $t_5 \ge t_4$ such that $\sigma(t, b) \ge t_4$ for $t \ge t_5$. Thus $z'[\sigma(t, b)] \ge z'(t)$, $t \ge t_5$. We have:

$$V'(t)\le -A(t)-\frac{\beta\sigma'(t,b)}{R^{\frac{1}{\beta}}(t)}\frac{1}{[z'(t)]^{(\alpha-\beta)/\beta}}V^{(\beta+1)/\beta}(t),\quad t\ge t_5$$

Since $z'(t)$ is a decreasing function for $t \ge t_4$, there exists a $T_2 \ge t_4$ such that $z'(t) \le z'(T_2)$ for $t \ge T_2$. We have:

$$V'(t)\le -A(t)-\frac{\beta\sigma'(t,b)}{R^{\frac{1}{\beta}}(t)}\frac{1}{[z'(T_2)]^{(\alpha-\beta)/\beta}}V^{(\beta+1)/\beta}(t),\quad t\ge t_5$$

Letting $k_2=\min\{1,\ [z'(T_2]^{(\beta-\alpha)/\beta}\}$, we get:

$$V'(t)\le -A(t)-\frac{\beta k_2\sigma'(t,b)}{R^{1/\beta}(t)}V^{(\beta+1)/\beta}(t),\quad t\ge t_5 \tag{6}$$

(II) If $\alpha \le \beta$, from *Lemma 1*, we get:

$$V'(t)=\frac{\{R(t)[z'(t)]^{\alpha}\}'}{z^{\beta}[\sigma(t,b)]}-\frac{\beta\sigma'(t,b)R(t)[z'(t)]^{\alpha}z'[\sigma(t,b)]}{z^{\beta+1}[\sigma(t,b)]}=$$

$$=\frac{\{R(t)[z'(t)]^{\alpha}\}'}{z^{\beta}[\sigma(t,b)]}-\frac{z'[\sigma(t,b)]}{z'(t)}\cdot\frac{\beta\sigma'(t,b)}{R^{\frac{1}{\alpha}}(t)}\{z[\sigma(t,b)]\}^{(\beta-\alpha)/\alpha}\left\{\frac{R(t)[z'(t)]^{\alpha}}{z^{\beta}[\sigma(t,b)]}\right\}^{(\alpha+1)/\alpha}$$

From *Lemma 1*, we get:

$$V'(t) \le -A(t) - \frac{z'[\sigma(t,b)]}{z'(t)} \frac{\beta\sigma'(t,b)}{R^{1/\alpha}(t)} \{z[\sigma(t,b)]\}^{(\beta-\alpha)/\alpha} V^{(\alpha+1)/\alpha}(t), \quad t \ge t_4$$

From *Lemma 1*, $\{R(t)[z'(t)^{\alpha}]\}' \le 0$, $t \ge t_4$. Since $\sigma(t, b) \ge t_4$ for $t \ge t_5$, hence:

$$R(t)[z'(t)]^{\alpha} \le R[\sigma(t,b)]\{z'[\sigma(t,b)]\}^{\alpha}, \quad t \ge t_5$$

That is:

$$\frac{z'[\sigma(t,b)]}{z'(t)} \ge \left\{\frac{R(t)}{R[\sigma(t,b)]}\right\}^{1/\alpha}, \quad t \ge t_5$$

We have:

$$V'(t) \le -A(t) - \left\{\frac{R(t)}{R[\sigma(t,b)]}\right\}^{1/\alpha} \frac{\beta\sigma'(t,b)}{R^{\frac{1}{\alpha}}(t)} \{z[\sigma(t,b)]\}^{(\beta-\alpha)/\alpha} V^{(\alpha+1)/\alpha}(t) =$$
$$= -A(t) - \frac{\beta\sigma'(t,b)}{R^{1/\alpha}[\sigma(t,b)]} \{z(\sigma[t,b))\}^{(\beta-\alpha)/\alpha} V^{(\alpha+1)/\alpha}(t), \quad t \ge t_5$$

Since $z'(t) > 0$ for $t \ge t_2$ and $[\mathrm{d}\sigma(t, \xi)]/\mathrm{d}t \ge 0$ for $t \ge t_0$, there exists a $T_3 \ge t_5$ such that $z[\sigma(t, b)] \ge z[\sigma(T_3, b)]$ for $t \ge T_3$. We have:

$$V'(t) \le -A(t) - \frac{\beta\sigma'(t,b)}{R^{1/\alpha}[\sigma(t,b)]} \{z[\sigma(T_3,b)]\}^{(\beta-\alpha)/\alpha} V^{(\alpha+1)/\alpha}(t), \quad t \ge t_5$$

Letting $k_3 = \min(1, \{z[\sigma(T_3, b)]^{(\beta-\alpha)/\alpha})$, we have:

$$V'(t) \le -A(t) - \frac{\beta k_3 \sigma'(t,b)}{R^{1/\alpha}[\sigma(t,b)]} V^{(\alpha+1)/\alpha}(t), \quad t \ge t_5 \tag{7}$$

We combine (I) and (II). Let $\lambda = \min\{\alpha, \beta\}$, $k_4 = \min\{k_2, k_3\}$, then eq. (6) combine with eq. (7) and write:

$$V'(t) \le -A(t) - \frac{\lambda k_4 \sigma'(t,b)}{R^{1/\lambda}[\sigma(t,b)]} V^{(\lambda+1)/\lambda}(t), \quad t \ge t_5$$

Letting:

$$B(t) = \frac{\lambda k_4 \sigma'(t,b)}{R^{1/\lambda}[\sigma(t,b)]}$$

we get:

$$V'(t) + A(t) + B(t) V^{(\lambda+1)/\lambda}(t) \le 0, \quad t \ge t_5$$

We consider set:

$$M = \{(t,s) : t \ge s \ge t_0\}, \quad M_0 = \{(t,s) : t > s \ge t_0\}$$

where $H(t, s) \in C(M, R)$ is said to belong to class X, write it as $H(t, s) \in X$, the following conditions are satisfied:

(i) $$H(t,t)=0,\ t\ge t_0,\ H(t,s)>0,\ (t,s)\in M_0$$

(ii) $$\frac{\partial H(t,s)}{\partial s}\le 0,\ \rho\in C^1([t_0,\infty),\mathrm{R}_+),\ h\in C^1(M_0,\mathrm{R})\ \text{and}$$

$$\frac{\partial H(t,s)}{\partial s}+\frac{\rho'(s)}{\rho(s)}H(t,s)=-h(t,s)H^{\frac{\lambda}{1+\lambda}}(t,s),\ (t,s)\in M_0$$

We give the following theorem by averaging of Philos type integrals.

Theorem 1. Assume that:

$$H\in\mathrm{X},\quad \rho\in C^1([t_0,\infty),\mathrm{R}_+),\ h\in C^1(M_0,\mathrm{R}),$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left(H(t,s)\rho(s)A(s)-\frac{\rho(s)R(\sigma[s,b)]\left|h(t,s)\right|^{\lambda+1}}{(\lambda+1)^{\lambda+1}[k_4\sigma'(s,b)]^{\lambda}}\right)\mathrm{d}s=\infty,\quad \forall T\ge t_7 \tag{8}$$

where

$$A(t)=kE(t)\int_a^b q(t,\xi)\left\{1-\int_c^d g[\sigma(t,\xi),u]\mathrm{d}u^{\beta}\right\}\mathrm{d}\xi,\quad k_4=\min\{k_2,\ k_3\}$$

$$k_2=\min\{1,\ [z'(T_2)]^{(\beta-\alpha)/\beta}\},\quad k_3=\min\{1,\ z[\sigma(T_3,b)]^{(\beta-\alpha)/\alpha}\}$$

$$\lambda=\min\{\alpha,\beta\}$$

then eq. (1) is oscillatory.

Proof. We use the method of proof to the contrary, suppose eq. (1) has non-oscillatory solutions $x(t)$, suppose $x(t) > 0$ for $t \ge t_0$. From *Lemma 2* we get:

$$V'(t)+A(t)+B(t)V^{(\lambda+1)/\lambda}(t)\le 0,\quad t\ge t_5$$

That is:

$$A(t)\le -V'(t)-B(t)V^{(\lambda+1)/\lambda}(t),\quad t\ge t_5$$

We have:

$$\int_T^t H(t,s)\rho(s)A(s)\mathrm{d}s\le -\int_T^t H(t,s)\rho(s)V'(s)\mathrm{d}s-\int_T^t H(t,s)\rho(s)B(s)V^{\frac{\lambda+1}{\lambda}}(s)\mathrm{d}s\le$$

$$\le H(t,T)\rho(T)V(T)+\int_T^t\left\{\left[\frac{\partial H(t,s)}{\partial s}+\frac{\rho'(s)}{\rho(s)}H(t,s)\right]\rho(s)V(s)-H(t,s)\rho(s)B(s)V^{(\lambda+1)/\lambda}(s)\right\}\mathrm{d}s\le$$

$$\le H(t,T)\rho(T)V(T)+\int_T^t[\left|h(t,s)\right|H^{\lambda/(\lambda+1)}(t,s)\rho(s)V(s)-H(t,s)\rho(s)B(s)V^{(\lambda+1)/\lambda}(s)]\mathrm{d}s,\ t\ge t_5$$

Using the inequality [13]:

$$Bu-Au^{(\lambda+1)/\lambda}\le\frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{B^{\lambda+1}}{A^{\lambda}},\quad \lambda>0,\quad A>0,\quad B\in\mathrm{R}$$

We have:

$$\int_T^t H(t,s)\rho(s)A(s)\mathrm{d}s \le H(t,T)\rho(T)V(T) + \int_T^t \frac{\lambda^\lambda \rho(s)|h(t,s)|^{\lambda+1}}{(\lambda+1)^{\lambda+1}B^\lambda(s)}\mathrm{d}s \le$$

$$\le H(t,T)\rho(T)V(T) + \int_T^t \frac{\rho(s)R(\sigma(s,b)|h(t,s)|^{\lambda+1}}{(\lambda+1)^{\lambda+1}[k_4\sigma'(s,b)]^\lambda}\mathrm{d}s, \quad t > T \ge t_5$$

That is:

$$\frac{1}{H(t,T)}\int_T^t \left\{ H(t,s)\rho(s)A(s) - \frac{\rho(s)R[\sigma(s,b)]|h(t,s)|^{\lambda+1}}{(\lambda+1)^{\lambda+1}[k_4\sigma'(s,b)]^\lambda} \right\}\mathrm{d}s \int_T^t \le \rho(T)V(T), \quad t > T \ge t_5$$

This formula is in contradiction with eq. (8), then eq. (1) is oscillatory.

Theorem 2. Hypothesis (2) holds and there exists a $\rho(t) \in C^1([t_0,\infty), R_+)$, such that:

$$\limsup_{t\to\infty} \int_{t_0}^t \left(\rho(s)A(s) - \frac{\lambda^\lambda[\rho'(s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\rho(s)B(s)]^\lambda} \right)\mathrm{d}s = \infty \tag{9}$$

where

$$A(t) = kE(t)\int_a^b q(t,\xi)\left\{1 - \int_c^d g[\sigma(t,\xi),u]\mathrm{d}u\right\}^\beta, \quad B(t) = \frac{\lambda k_4\sigma'(t,b)}{R^{1/\lambda}[\sigma(t,b)]}, \quad \lambda = \min\{\alpha,\beta\}$$

then eq. (1) is oscillatory.

Proof. From eq (5) we have:

$$\int_T^t \rho(s)A(s)\mathrm{d}s \le -\int_T^t \rho(s)V'(s)\mathrm{d}s - \int_T^t \rho(s)B(s)V^{\frac{\lambda+1}{\lambda}}(s)\mathrm{d}s =$$

$$= -\rho(t)V(t) + \rho(T)V(T) + \int_T^t [\rho'(s)V(s) - \rho(s)B(s)V^{\frac{\lambda+1}{\lambda}}(s)]\mathrm{d}s$$

Using the inequality [13]:

$$Bu - Au^{(\lambda+1)/\lambda} \le \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\frac{B^{\lambda+1}}{A^\lambda}, \quad \lambda > 0, \quad A > 0, \quad B \in \mathrm{R}$$

We have:

$$\int_T^t \rho(s)A(s)\mathrm{d}s \le \rho(T)V(T) + \int_T^t \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\frac{[\rho'(s)]^{\lambda+1}}{[\rho(s)B(s)]^\lambda}\mathrm{d}s, \quad t > T \ge t_5$$

That is:

$$\int_T^t \left\{ \rho(s)A(s) - \frac{\lambda^\lambda}{(\lambda+1)^{\lambda+1}}\frac{[\rho'(s)]^{\lambda+1}}{[\rho(s)B(s)]^\lambda} \right\}\mathrm{d}s \le \rho(T)V(T), \quad t > T \ge t_5$$

This formula is in contradiction with eq. (9), then eq. (1) is oscillatory.

Theorem 3. The following is satisfied:

$$\liminf_{t\to\infty}\frac{1}{G(t)}\int_t^\infty B(s)G^{(\lambda+1)/\lambda}(s)\mathrm{d}s>\frac{\lambda}{(\lambda+1)^{(\lambda+1)/\lambda}},\quad \forall t\ge t_0 \tag{10}$$

where

$$G(t)=\int_t^\infty A(s)\mathrm{d}s,\quad B(t)=\lambda k_4\sigma'(t,b)R^{-1/\lambda}[\sigma(t,b)],\quad \lambda=\min\{\alpha,\beta\},\quad k_2=\min\{1,\ [z'(T_2)]^{(\beta-\alpha)/\beta}\}$$

$$k_3=\min(1,\ \{z[\sigma(T_3,b)]\}^{(\beta-\alpha)/\alpha})\ ,\quad k_4=\min\{k_2,\ k_3\}$$

$$A(t)=kE(t)\int_a^b q(t,\xi)\left\{1-\int_a^b g[\sigma(t,\xi),u]\mathrm{d}u\right\}^\beta \mathrm{d}\xi$$

then eq. (1) is oscillatory.

Proof. Suppose that eq. (1) has a non-oscillatory solutions $x(t)$, without losing generality, assume that $x(t)>0$ for $t\ge t_0$. From *Lemma 2* we have:

$$\int_t^\infty V'(s)\mathrm{d}s+\int_t^\infty A(s)\mathrm{d}s+\int_t^\infty B(s)V^{(\lambda+1)/\lambda}(s)\mathrm{d}s\le 0,\quad t\ge t_5$$

That is:

$$\int_t^\infty A(s)\mathrm{d}s+\int_t^\infty B(s)V^{(\lambda+1)/\lambda}(s)\mathrm{d}s\le V(t),\quad t\ge t_5$$

Letting:

$$G(t)=\int_t^\infty A(s)\mathrm{d}s$$

hence:

$$\frac{V(t)}{G(t)}\ge 1+\frac{1}{G(t)}\int_t^\infty B(s)V^{(\lambda+1)/\lambda}(s)\mathrm{d}s,\quad t\ge t_5 \tag{11}$$

From eq. (10), there exists $\delta\ge\lambda/[(1+\lambda)^{\lambda+1}]$ such that:

$$\liminf_{t\to\infty}\frac{1}{G(t)}\int_t^\infty B(s)G^{(\lambda+1)/\lambda}(s)\mathrm{d}s>\delta>\frac{\lambda}{(\lambda+1)^{(\lambda+1)/\lambda}},\quad t\ge t_5 \tag{12}$$

Letting:

$$L=\inf_{t\ge t_5}\left\{\frac{V(t)}{G(t)}\right\}$$

from (11), we have $L \geq 1$, thus:

$$\frac{V(t)}{G(t)} - \frac{1}{G(t)}\int_t^{\infty} B(s)V^{(\lambda+1)/\lambda}(s)\mathrm{d}s \geq 1, \quad t \geq t_5$$

We have:

$$\liminf_{t\to\infty}\frac{V(t)}{G(t)} - \liminf_{t\to\infty}\frac{1}{G(t)}\int_t^{\infty} B(s)G^{\lambda/(\lambda+1)}(s)\left[\frac{V(s)}{G(s)}\right]^{(\lambda+1)/\lambda}\mathrm{d}s, \quad t \geq t_5$$

and:

$$L - \delta L^{(\lambda+1)/\lambda} \geq 1 \tag{13}$$

From eq. (12), we have:

$$\delta > \frac{\lambda}{(\lambda+1)^{(\lambda+1)/\lambda}}$$

That is:

$$\delta^{\lambda} > \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}$$

Using the inequality [13]:

$$Bu - Au^{(\lambda+1)/\lambda} \leq \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{B^{\lambda+1}}{A^{\lambda}}, \quad \lambda > 0, \quad A > 0, \quad B \in \mathrm{R}$$

We have:

$$L - \delta L^{(\lambda+1)/\lambda} \leq \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{1}{\delta^{\lambda}} < \frac{\lambda^{\lambda}}{(\lambda+1)^{\lambda+1}}\frac{(\lambda+1)^{\lambda+1}}{\lambda^{\lambda}} = 1$$

This formula is in contradiction with eq. (13), then eq. (1) is oscillatory.

Example. Consider the following differential equation:

$$\{\varphi_{\alpha}[z'(t)]\}' + \frac{1}{t}\varphi_{\alpha}[z'(t)] + \int_a^b \frac{\ln\xi}{\xi^{1+\lambda}}\varphi_{\beta}\left[x\left(\tfrac{t-\xi}{2}\right)\right]\mathrm{d}\xi = 0 \tag{14}$$

where $\varphi_{\lambda} = |s|^{\lambda-1}s$, $1 < a < b$, $\alpha > 1$, $\beta > 0$.

Letting:

$$z(t) = x(t) + \int_1^2 \frac{1}{2}x(t-1)\mathrm{d}\xi$$

we have:

$$r(t) = 1, \quad \tau(t,\xi) = t-1, \quad \sigma(t,\xi) = \frac{t-\xi}{2}, \quad p(t) = \frac{1}{t}, \quad q(t,\xi) = \frac{\ln\xi}{\xi^{1+\lambda}}$$

$$g(t,\xi) = \frac{1}{2}, \quad \lambda = \min\{\alpha,\beta\}$$

Letting $t_0 = 1$, $\rho(t) = t^\lambda$, we get:

$$E(t)=t, \quad R(t)=t, \quad \Phi(t)=\int_t^\infty R^{-1/\alpha}(s)\mathrm{d}s=\int_t^\infty s^{-1/\alpha}\mathrm{d}s=\infty$$

$$A(t)=\frac{kt}{2^\beta}\int_a^b \frac{\ln\xi}{\xi^{1+\lambda}}\mathrm{d}\xi, \quad B(t)=\frac{\lambda k_4}{2^{1-1/\lambda}(t-b)^{\frac{1}{\lambda}}}$$

$$\limsup_{t\to\infty}\int_{t_0}^t\left\{\rho(s)A(s)-\frac{\lambda^\lambda[\rho'(s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[\rho(s)B(s)]^\lambda}\right\}\mathrm{d}s=$$

$$=\limsup_{t\to\infty}\int_1^t\left\{s^\lambda\frac{ks}{2^\beta}\int_a^b\frac{\ln\xi}{\xi^{1+\lambda}}\mathrm{d}\xi-\frac{\lambda^\lambda(\lambda s^{\lambda-1})^{\lambda+1}}{(\lambda+1)^{\lambda+1}\left[s^\lambda\dfrac{\lambda k_4}{2^{1-1/\lambda}(s-b)^{\frac{1}{\lambda}}}\right]^\lambda}\right\}\mathrm{d}s=$$

$$=\limsup_{t\to\infty}\int_1^t\left[s^{\lambda+1}\frac{k}{2^\beta}\int_a^b\frac{\ln\xi}{\xi^{1+\lambda}}\mathrm{d}\xi-\frac{2^{\lambda-1}\lambda^{\lambda+1}}{k_4^\lambda(\lambda+1)^{\lambda+1}}\frac{s-b}{s}\right]\mathrm{d}s=\infty$$

Then the conditions of *Theorem 2* are satisfied, then eq. (14) is oscillatory.

Conclusion

In the domain of research on second-order non-linear differential equations with damping terms, which play a pivotal role in the fields of physics and engineering, a substantial body of research has been dedicated to the exploration of their oscillatory behavior [24-27]. This paper undertakes an investigation into the oscillation of a class of second-order non-linear differential equations with damping terms that exert substantial influence in the aforementioned disciplines. The application of the generalized Riccati transform technique, in conjunction with specific methodologies, has led to the establishment of oscillation criteria for differential equations.

Acknowledgment

Xi Liu and Jun Liu are common first authors. This work was supported by Chinese Natural Science Foundation Grant No. 62462056, the Yunnan Fundamental Research Projects under Grant No 202501AT070393, the Special Basic Cooperative Research Innovation Programs of Qujing Science and Technology Bureau & Qujing Normal University Grant No. KJLH2023YB12,the Basic Research Joint Project of Local Undergraduate Universities in Yunnan Province Grant No. 202401BA070001-135,the Scientific Research Foundation of the Yunnan Provincial Department of Education Grant No. 2024J0932.

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Paper submitted: June 30, 2024
Paper revised: August 15, 2025
Paper accepted: August 16, 2025