A NEW METHOD FOR NUMERICAL SOLUTION OF WAVE EQUATION IN HYPERBOLIC MODEL BASED ON STOCHASTIC SIMULATION

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The objective of this paper is to examine the numerical solution of the wave equation. The wave equation is discretized by means of the implicit difference method, thereby yielding a large sparse system of linear algebraic equations (AU = b). Subsequently, the Jacobi over-relaxation iterative method is employed to transform it into the form of U = LU + f. The Monte Carlo method is employed to solve this system of equations. A particular instance substantiates the efficacy of this approach in approximating the exact solution with a reasonable degree of accuracy when solving the numerical solution of the wave equation, thus offering a novel methodology for the numerical solution of hyperbolic models.

Key words: *wave equation, Monte Carlo random sampling method, implicit difference method*

Introduction

Partial differential equations (PDE) are ubiquitous in a wide array of scientific and engineering disciplines. In the field of condensed matter physics, they govern the behavior of electrons and lattice vibrations. In fluid mechanics, they regulate the flow of fluids, a crucial aspect of applications in aviation, meteorology, and oceanography; and in optics, they elucidate the propagation of light. The procurement of both exact and numerical solutions to PDE is of paramount importance in a variety of fields, including engineering design and scientific research.

It is evident that, over the course of many years, a considerable number of researchers have dedicated themselves to this field of study. Consequently, a substantial number of methods have been developed. Notable achievements include the development of the homotopy perturbation method [1-4]. This approach utilizes homotopy concepts to perturb the original equation, thereby facilitating the approximation of solutions. The variational iteration method [5, 6] integrates variational principles with iteration techniques, offering an effective approach to solving non-linear PDE. The Taylor series method [7] utilizes the Taylor series expansion to approximate solutions, providing a straightforward yet powerful tool for numerical analysis.

The point-solution method [8] prioritizes the identification of highly precise solutions at designated points. This targeted approach can be particularly useful in scenarios

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where local accuracy is of utmost importance. The modified Old Babylonian algorithm [9] provides a novel perspective. A reevaluation of ancient ideas through the lens of modern mathematics demonstrates the potential for the development of innovative and promising methods. The exp function method [10] has been demonstrated to be a highly effective tool for the analysis of non-linear wave-type PDE. This method facilitates the management of complex wave-related problems by researchers, enabling them to perform their analyses with relative ease.

Finally, variational-based methods [11-13] are predicated on variational formulations as a means to address PDE. These methodologies provide a rigorous and systematic framework for analyzing and solving PDE, often yielding accurate and reliable results. Each of these methods possesses unique characteristics and applications, thereby contributing significantly to the advancement of the field.

The present article focuses on a distinct numerical approach for PDE: the Monte Carlo stochastic simulation method. Monte Carlo methods have been shown to possess several advantageous properties [14]. Firstly, it is important to note that they are parallel algorithms. This characteristic enables high-efficiency parallel computing, which is highly beneficial for handling computationally intensive tasks. Secondly, these systems are well suited for solving large sparse systems of linear algebraic equations. Such systems frequently emerge when approximating PDE. The discretization process often leads to a large number of linear equations with a sparse coefficient matrix structure.

In the domain of fluid-related fields, such as aviation, meteorology, oceanography, and water conservancy, the utilization of hyperbolic equations is a prevalent practice for the modeling of diverse physical phenomena. In this paper, we employ the wave equation as the research model and explore the following mixed initial-boundary value problem [15]:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad 0 < x < 1, \quad 0 < t \le T \tag{1}$$

with initial condition:

$$u(x,0) = \varphi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \psi(x), \qquad 0 \le x \le 1 \tag{2}$$

and boundary conditions:

$$u(0,t) = \alpha(t), \qquad u(1,t) = \beta(t), \qquad 0 < t \le T \tag{3}$$

where a is a positive constant and $f(x,t), \varphi(x), \psi(x), \alpha(t), \beta(t)$ – the known continuous functions. This study aims to apply the Monte Carlo stochastic simulation method to this problem and evaluate its effectiveness in obtaining numerical solutions.

Implicit difference method for discretizing

The domain $[0,1]\times[0,T]$ is divided into an $m \times n$ mesh with the spatial-step size $h = 1/m$ in the x-direction and the time-step size $\tau = T/n$, respectively. Grid points (x_i, t_k) are defined by:

$$x_i = ih, \quad i = 1, 2, \cdots, m$$

$$t_k = k\tau, \quad k = 1, 2, \cdots, n$$

For eqs. (1)-(3), the following difference scheme can be established [15]:

$$-\frac{1}{2}s^2u_{i-1}^{k+1}+(1+s^2)u_i^{k+1}-\frac{1}{2}s^2u_{i+1}^{k+1}=\frac{1}{2}s^2u_{i-1}^{k-1}-(1+s^2)u_i^{k-1}+$$
$$+\frac{1}{2}s^2u_{i+1}^{k-1}+2u_i^k+\tau^2f(x_i,t_k) \tag{4}$$

where $s=a\tau/h$, $1\le i\le m-1, 1\le k\le n-1$.

Equation (4) can be written as the following Matrix form:

$$AU=b \tag{5}$$

where

$$A=\begin{bmatrix} 1+s^2 & -\frac{1}{2}s^2 & & & \\ -\frac{1}{2}s^2 & 1+s^2 & -\frac{1}{2}s^2 & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{1}{2}s^2 & 1+s^2 & -\frac{1}{2}s^2 \\ & & & -\frac{1}{2}s^2 & 1+s^2 \end{bmatrix}$$

$$\bar{b}=\begin{bmatrix} -(1+s^2) & \frac{1}{2}s^2 & & & \\ \frac{1}{2}s^2 & -(1+s^2) & \frac{1}{2}s^2 & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2}s^2 & -(1+s^2) & \frac{1}{2}s^2 \\ & & & \frac{1}{2}s^2 & -(1+s^2) \end{bmatrix}$$

$$b=\bar{b}\begin{bmatrix} u_1^{k-1} \\ u_2^{k-1} \\ \vdots \\ u_{m-2}^{k-1} \\ u_{m-1}^{k-1} \end{bmatrix}+\begin{bmatrix} \frac{1}{2}s^2(u_0^{k+1}+u_0^{k-1})+2u_1^k+\tau^2f(x_1,t_k) \\ 2u_2^k+\tau^2f(x_2,t_k) \\ \vdots \\ 2u_{m-2}^k+\tau^2f(x_{m-2},t_k) \\ \frac{1}{2}s^2(u_m^{k+1}+u_m^{k-1})+2u_{m-1}^k+\tau^2f(x_{m-1},t_k) \end{bmatrix}, \quad 1\le k\le n-1$$

The solution of the linear system of algebraic equation

Consider the following Jacobi over-relaxation iterative method for solving (5):

$$u_i^{(k)} = (1-\gamma)u_i^{(k-1)} + \frac{\gamma}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}u_j^{(k-1)} - \sum_{j=i+1}^{n-1} a_{ij}u_j^{(k-1)}\right] \tag{6}$$

where $i = 1, 2, \cdots, n-1$ and $\gamma \in (0,1]$.

Equation (6) can be written:

$$U^{(k)} = LU^{(k-1)} + f \, , k = 1, 2, \cdots \tag{7}$$

where $U^{(k)} = (u_1^{(k)}, u_2^{(k)}, \cdots, u_{n-1}^{(k)})^T$, $L = I - DA$, $f = Db$, and:

$$D = diag\left(\frac{\gamma}{a_{11}}, \cdots, \frac{\gamma}{a_{n-1,n-1}}\right) \tag{8}$$

is a diagonal matrix. Now, system (5) can be presented in the following form:

$$U = LU + f \tag{9}$$

In fact, A is a diagonally dominant matrix, then the elements of matrix L must satisfy the following condition:

$$\sum_{j=1}^{n-1} | l_{ij} | < 1, \quad i = 1, 2, \cdots, n-1 \tag{10}$$

Let $U^{(0)} = 0$, from (7), we have:

$$U^{(k)} = (I + L + \cdots + L^{k-1}) f = \sum_{m=0}^{k-1} L^m f \, , \quad k = 1, 2, \cdots \tag{11}$$

It is well known that (10) is a sufficient condition for convergence of (11), *i.e.*:

$$\lim_{k\to\infty} U^{(k)} = U$$

In the next section we compute the iterations $U^{(k)}$ using Monte Carlo stochastic simulation method.

Monte Carlo method to solve linear system of algebraic equations

The application of Monte Carlo method to find a solution of system (9) is as follows: Consider the Markov chain:

$$x_0 \to x_1 \to x_2 \to \cdots \to x_k \to \cdots \tag{12}$$

with state space $\{1, 2, \cdots, n-1\}$ and transition matrix $P = \{p_{ij}\}$, $i, j = 1, \cdots, n-1$. Let:

$$P(x_0 = i) = p_i \, , \quad P(x_{n-1} = j \mid x_{n-2} = i) = p_{ij} \tag{13}$$

where p_i, p_{ij} are the initial distribution and transition probabilities of the Markov chain.

The weight function W_m for the Markov chain (12) is defined in the following form:

$$W_0 = 1, \quad W_m = W_{m-1}\frac{l_{x_{m-1}x_m}}{p_{x_{m-1}x_m}}, \quad m = 1, 2, \cdots$$

Random variable associated with the following sample path:

$$x_0 \to x_1 \to x_2 \to \cdots \to x_k$$

is defined:

$$\Gamma_k[H] = \frac{H_{x_0}}{p_{x_0}}\sum_{m=0}^{k} W_m f(b_{x_m})$$

where k is a integer number and $H^t = (h_1, ..., h_{n-1})$ is given vector.

Theorem 1. The mathematical expectation value of the random variable $\Gamma_k[H]$ is equal to the inner product:

$$\langle H, U^{(k)}\rangle, \textit{ i.e., } E(\Gamma_k[H]) = \langle H, U^{(k)}\rangle$$

To estimate:

$$\langle H, U^{(k)}\rangle = h_1 u_1^{(k)} + ... + h_{n-1}u_{n-1}^{(k)}$$

where

$$U^{(k)} = \left(u_1^{(k)}, u_2^{(k)}, \cdots, u_{n-1}^{(k)}\right)$$

is the k^{th} iterative solution of (9), we simulate N random paths:

$$x_0^{(s)} \to x_1^{(s)} \to x_2^{(s)} \to \cdots \to x_k^{(s)}, \, s = 1, \cdots, N$$

each with the length k, and evaluate the sample mean:

$$\langle H, U^{(k)}\rangle \approx \frac{1}{N}\sum_{m=1}^{k}\Gamma_k^{(s)}[H]$$

It is seen that by setting:

$$H^t = (\underbrace{0, ..., 1}_{j}, 0, ..., 0)$$

we have:

$$\langle H, U^{(k)}\rangle = u_j^{(k)}, \, j = 1, ..., n-1$$

therefore, $\Gamma_k[H]$ is an unbiased estimator of the $u_j^{(k)}$.

Discussion

In this section, we perform numerical tests for the algorithm described in previous sections.

Example 1. [15] Consider (1)-(3) with the following given conditions:

$$a=1, \quad f(x,t)=0, \quad u(x,0)=e^x, \quad \frac{\partial u}{\partial t}(x,0)=e^x, \quad 0\le x\le 1,$$
$$u(0,t)=e^t, \quad u(1,t)=e^{1+t}, \quad 0<t\le 1,$$

This example admits the exact solution is $u(x,t)=e^{x+t}$.

The results obtained for $u(x, t)$ with $\gamma = 0.98$, $h = 0.01$, $\tau = 0.01$, $k = 17$, and $N = 5000$ are presented in tab. 1.

Table 1. The γ = 0.98, h = 0.01, τ = 0.01, k = 17, N = 5000

(x, t)	Numerical	Exact	| Exact-Numerical|
(0.5,0.1)	1.821947	1.822119	$1.72\cdot10^{-4}$
(0.5,0.2)	2.012607	2.013753	$1.15\cdot10^{-3}$
(0.5,0.3)	2.223287	2.225541	$2.25\cdot10^{-3}$
(0.5,0.4)	2.456001	2.459603	$3.60\cdot10^{-3}$
(0.5,0.5)	2.713645	2.718282	$4.64\cdot10^{-3}$
(0.5,0.6)	2.997337	3.004166	$6.83\cdot10^{-3}$
(0.5,0.7)	3.310889	3.320117	$9.23\cdot10^{-3}$
(0.5,0.8)	3.657673	3.669297	$1.16\cdot10^{-2}$
(0.5,0.9)	4.041330	4.055200	$1.39\cdot10^{-2}$
(0.5,1.0)	4.467903	4.481689	$1.38\cdot10^{-2}$

Conclusions

In this paper, we have introduced a novel approach for solving hyperbolic problems, which significantly differs from those in open literature. The proposed methodology is initiated with the discretization of the governing equations through the implementation of the implicit difference method. This step is of paramount importance because it converts continuous-domain PDE into a discrete form of a large sparse system of linear algebraic equations. This conversion renders the equations more amenable to numerical computations.

Subsequently, the Jacobi over-relaxation iterative method is employed. This iterative technique not only simplifies the structure of the algebraic equations but also provides a more efficient way to approach the solution. The conversion of the system into the form $U = LU + f^6$ establishes the foundation for the implementation of the Monte Carlo method.

The Monte Carlo method is then employed to solve the resulting algebraic equations, utilizing a unique strategy based on random sampling. This approach leverages the parallel nature of Monte Carlo algorithms, rendering it well-suited for the management of large-scale problems.

In order to validate the performance of the proposed method, a numerical experiment was conducted on a specific wave equation-based example. As demonstrated in tab. 1, the results exhibit a high degree of accuracy in estimating the exact solutions of eqs. (1)-(3). The modest discrepancies between the numerical and exact solutions, particularly in light of the intricacy of the problem and the simplicity of the proposed method, are encouraging.

However, it is imperative to acknowledge the limitations of this study. The theoretical analysis could be further developed. For instance, further research is necessary to explore the convergence rate of the Jacobi over-relaxation iterative method and the Monte Carlo method in this context. Furthermore, the potential for the application of this method remains to be expanded. Future research endeavors could explore its effectiveness for different types of hyperbolic equations and in more complex physical scenarios.

In summary, despite its limitations, the proposed method offers a novel approach and a practical alternative for solving hyperbolic problems. The model has the potential for further development and refinement for broader applications in various scientific and engineering fields where hyperbolic models are widely used.

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