INVESTIGATION OF CLOSED FORM SOLITARY WAVE SOLUTIONS FOR THE COUPLED NON-LINEAR REACTION-DIFFUSION MODEL

by

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In this study, the coupled non-linear reaction-diffusion model is under consideration for the exact solitary wave solutions. These equations are visible in a vast range of conditions, including not only biological and chemical and physical disciplines, as well as in the environmental and social sciences. The well-known approach namely as generalized Riccati equation mapping method (GREM) is employed to produce closed form exact solitary wave solutions to a particular coupled non-linear diffusion reaction equations. After applying this method, we perceive analytical solutions in the form of exponential, trigonometric, rational and hyperbolic functions. Moreover, for the physical interpretation, some solutions are drawn in the form of 3-D, 2-D, and corresponding contours for the different values of parameters.

Key words: *coupled system, exact solutions, non-linear diffusion equation generalized Riccati equation mapping method*

Introduction

A compulsory part of researching non-linear wave scenarios is studying the accurate solutions of non-linear PDE. Non-linear wave phenomena take an important role in both our everyday lives and a variety of study fields including fluid mechanics [1], electromagnetism [2], plasma physics [3], non-linear optics [4], optical fibers [5], fluid dynamics [6, 7], quantum mechanics [8], and many others. The physical occurrences are more carefully reported by the

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non-linear PDE. With the use of mathematical instruments like MAPLE, MATHEMATICA, and MATLAB, which make finishing difficult and too-long algebraic calculations easier. It has become more important recently to find out the correct numerical and analytical solutions for non-linear PDE. Many mathematicians have generated techniques. For finding non-linear PDE solutions such as the jacobi elliptic function method [9], the exponential function expansion method [10], rational homotopy perterbation method [11], the generalized Riccati equation mapping method [12], f 6 model expansion method [13, 14], the homogeneous balance method [15], the tanh-method [16], the inverse scattering transform [17], the backlund transform [18], the modified extended fan subequation method [19], the truncated painleve expansion [20], and the auxiliary equation method [21], using the generalized exponential rational function method [22-25], for the various non-linear PDE. Ghanbari, *et al.* [22-25] explored the numerous kinds of hyperbolic, exponential, trignometric and soliton solutions. To study (2+1) dimentional Boiti-Leon-Peninelle equation Zhu initiated the generalized Riccati equation mapping with the extended tanh-function. The GREM is a strong analytical method for solving a variety of differential equations mostly non-linear ones. For resolving an ample of non-linear differential equations the GREM is a precious analytical implement. Its benefits comprise handling, non-linearity, generality and the capability to simplify and resolve issues. When selecting whether to take on this method,researchers should think about their experience with GREM, the features of the problem, and the required level of analytical rigor. There are many real-world uses for non-linear PDE that are really remarkable and relevent to real-world phenomena. The meteorological report you see on TV each night is taken from the numerical part of a complex set of non-linear polynomials, and dynamic meteorology and numerical weather predicting are obvious example.

The novelty and motivation of this study are that we obtained the exact solitary wave solutions for the coupled reaction-diffusion model. These equations are visible in a vast range of conditions, including not only in biological chemical, and physical disciplines but as well as in the environmental and social sciences. The GREM method is used to obtain these solutions. The different forms of exact solitary wave solutions are extracted in the form of exponential, trigonometric, rational, and hyperbolic functions. For, the physical interpretation of these results we dram some solutions in the form of 3-D, 2-D, and corresponding contour plots. So, these results are very effective for the reaction-diffusion coupled model when these waves are traveling from one place to another place.

Model discription

A diversity of prey-predator situations are modeled utilizing the Lotka-Voltera model and its variations. Remarkably, the population flux between urban and rural areas is also estimated by implementing the same set of equations, essentially through analogy. The non-linear relation between the two populations in this model produces two set of coupled PDE. It will be feasible to explore the solution dependency on the suitable parameters in a way that is more definite if the analytical solution these equation becomes approachable. For nearly eight decades, this system has aroused curiosity, but its modified kind with diffusion terms has only been considered through empirical research. Some coupled diffusion reaction equations of a properly general kind will be looked into the latest work.The following coupled DR-equations [26] will be exactly solved utilizing the Riccati method in this case:

$$\begin{aligned} u_t - v_1 u_x &= d_1 u_{xx} + l_1 u^2 v \\ v_t - v_2 v_x &= l_2 \mu v^2 - d_2 v u_{xx} \end{aligned} \tag{1}$$

The condition in which both the predator and the prey seprate by diffusion is related to these two sets of equations. Specifically, eq. (1) emerges when the diffusion coefficients becomes density-dependent as indicated by the diffusion terms, d_1u and d_2v, coefficients. In this equation v and u are the populations of the predator and the prey. Here are the positive constants, l_1 and l_2. The diffusion coefficients are d_1 and d_2. Moreover, the prey and the predator convective velocities are v_1 and v_2.

The generalized Riccati equation mapping mathod

The generalized Riccati equation mapping method's fundamental concept is that for a given non-linear partial diffrential equation (NPDE) with independent variable $x = (x_0 = x_1, x_2,..., x_t)$, and dependent variable, w:

$$P\left(w, w_t, w_{x_i}, w_{x_i x_j}, ...\right) = 0 \tag{2}$$

where P is a polynomial function of its argument and the following anstaz is acquired from eq. (1) when the subscripts demote the partial derivative using the transformation:

$$w = w(\chi), \;\; \chi = \chi(x_1, x_2, ...) \tag{3}$$

where χ is real function that requires to be resolved. An ODE is produced by inserting eq. (3) in eq. (2):

$$O\left(w, w_\chi, w_{\chi\chi}, ...\right) = 0 \tag{4}$$

We consider that solution of eq. (4) is in the type:

$$w(\chi) = \sum_{k=0}^{m} \left(a_k \psi \chi\right)^k \tag{5}$$

where a_k is the function of (x, v) to be resolved. In order to show the solution of the following summarized Riccati equation, $\psi(\chi)$ must balance the highest order linear term with the non-linear term. This fixes m:

$$\psi'(\chi) = f + d\psi(\chi) + e\psi^2(\chi) \tag{6}$$

where f, d, and e are the variable real constant. Put eq. (5) and eq. (6) into a relevant ODE eq. (4) and ending all the coefficient of ψ will gives a system of algebric equations, from these equation we get the parameters a_1, $(k = 1,..., m)$ and χ. Finding the non-travelling wave solution NPDE is a simple process when using the known solutions of the algabric equations.

These nine solution of equation eq. (6) can be acquired:

- *Type 1*: When $d^2 - 4ef > 0$ and $de \neq 0$ or $(ef \neq 0)$, here $(d^2 - 4ef)^{1/2} = \alpha_1$

$$\psi_1 = -\frac{1}{2e}\left[d + \alpha_1\left(\tanh\left(\alpha_1\chi\right) \pm \operatorname{sech}\left(\alpha_1\chi\right)\right)\right] \tag{7}$$

$$\psi_2 = -\frac{1}{4e}\left[2d + \alpha_1\left(\tanh\left(\frac{\alpha_1}{4}\chi\right) \pm \coth\left(\frac{\alpha_1}{4}\chi\right)\right)\right] \tag{8}$$

$$\psi_3 = \frac{1}{2e}\left[-d + \frac{\sqrt{\left(C^2 + D^2\right)}\alpha_1 - C\alpha_1\cosh\left(\alpha_1\chi\right)}{C\sinh\left(\alpha_1\chi\right) + D}\right] \tag{9}$$

where C and D are the two non-zero real constants and fulfill $D^2 - C^2 > 0$.

– *Type 2*: When $d^2 - 4ef > 0$ and $de \neq 0$ or $(ef \neq 0)$, here $(4ef - d^2)^{1/2} = \alpha_2$

$$\psi_4 = \frac{1}{2e}\left[-d + \alpha_2\left(\tan\left(\alpha_2\chi\right) \pm \sec\left(\alpha_2\chi\right)\right)\right] \tag{10}$$

$$\psi_5 = \frac{1}{4e}\left[-2d + \alpha_2\left(\tan\left(\frac{\alpha_2}{4}\chi\right) - \cot\left(\frac{\alpha_2}{4}\chi\right)\right)\right] \tag{11}$$

$$\psi_6 = \frac{2f\sin\left(\frac{\alpha_2}{2}\chi\right)}{-d\sin\left(\frac{\alpha_2}{2}\chi\right) + \alpha_2\cos\left(\frac{\alpha_2}{2}\chi\right)} \tag{12}$$

$$\psi_7 = -\frac{2f\sin\left(\frac{\alpha_2}{2}\chi\right)}{-d\sin\left(\alpha_2\chi\right) + \alpha_2\cos\left(\alpha_2\chi\right) \pm \alpha_2} \tag{13}$$

– *Type 3*: When $e \neq 0$ and $d = f = 0$

$$\psi_8 = -\frac{1}{e\chi + h_1} \tag{14}$$

where h_1 is the arbitrary constant.

– *Type 4*: When $f = 0$ and $de \neq 0$.

$$\psi_9 = -\frac{\left[\cosh\left(d\chi\right) + \sinh\left(d\chi\right)\right]}{e\left[g + \cosh\left(d\chi\right) + \sinh\left(d\chi\right)\right]} \tag{15}$$

where g is the arbitrary constant.

Exact solitary wave solutions

Now transform PDE of eq. (1) into ODE using transform:

$$u\left(x,t\right) = u\left(\chi\right) \text{ where } \chi = x - \omega t \tag{16}$$

by substituting this transformation into eq. (1) we get:

$$\begin{gathered}\left(v_1 + \omega\right)u' - d_1 u v'' - l_1 u^2 v = 0 \\ \left(v_2 + \omega\right)v' + d_2 v u'' - l_2 v^2 u = 0\end{gathered} \tag{17}$$

Now, we suppose the general solution of eq. (20) in the form of polynomials:

$$\begin{gathered}u\left(\chi\right) = \sum_{l=0}^{M} m^i z^i \\ v\left(\chi\right) = \sum_{\iota=0}^{M} n^i z^i\end{gathered} \tag{18}$$

Applying the balancing procedure to eq. (20) gives us $M = 2$, get:

$$\begin{gathered}u\left(\chi\right) = m_0 + m_1 z\left(\chi\right) + m_2 z(\chi)^2 \\ v\left(\chi\right) = n_0 + n_1 z\left(\chi\right) + n_2 z(\chi)^2\end{gathered} \tag{19}$$

where m_0, m_1, m_2, n_0, n_1, and n_2 are the constants here.

Using mathematica by solving these systems of equations we get the following unknown like:

$$\begin{gathered}\omega=-v_2,\ v_1=v_2,\ m_0=-\frac{d_1\left(d^2+2ef\right)}{l_1},\ m_1=-\frac{6dd_1e}{l_1}\\ m_2=-\frac{6d_1e^2}{l_1},\ n_0=-\frac{6efd_2}{l_2},\ n_1=-\frac{6dd_2e}{l_2},\ n_2=-\frac{6d_2e^2}{l_2}\end{gathered}\tag{20}$$

- *Type 1*: When $d^2-4ef>0$ and $de\neq 0$ or $ef\neq 0$.

$$\begin{gathered}u(x,t)=-\frac{d_1\left(d^2+2ef\right)}{l_1}+\frac{3dd_1\left(\alpha_1\left(\tanh\left(\alpha_1\left(tv_2+x\right)\right)+\operatorname{sech}\left(\alpha_1\left(tv_2+x\right)\right)\right)+d\right)}{l_1}-\\ -\frac{3d_1\left(\alpha_1\left(\tanh\left(\alpha_1\left(tv_2+x\right)\right)+\operatorname{sech}\left(\alpha_1\left(tv_2+x\right)\right)\right)+d\right)^2}{2l_1}\\ v(x,t)=-\frac{6efd_2}{l_2}+\frac{3dd_2\left(\alpha_1\left(\tanh\left(\alpha_1\left(tv_2+x\right)\right)+\operatorname{sech}\left(\alpha_1\left(tv_2+x\right)\right)\right)+d\right)}{l_2}-\\ -\frac{3d_2\left(\alpha_1\left(\tanh\left(\alpha_1\left(tv_2+x\right)\right)+\operatorname{sech}\left(\alpha_1\left(tv_2+x\right)\right)\right)+d\right)^2}{2l_2}\end{gathered}\tag{21}$$

$$\begin{gathered}u(x,t)=-\frac{d_1\left(d^2+2ef\right)}{l_1}+\frac{3dd_1\left(\alpha_1\tanh\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+\coth\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+2d\right)}{2l_1}-\\ -\frac{3d_1\left(\alpha_1\tanh\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+\coth\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+2d\right)^2}{8l_1}\\ v(x,t)=-\frac{6efd_2}{l_2}+\frac{3dd_2\left(\alpha_1\tanh\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+\coth\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+2d\right)}{2l_2}-\\ -\frac{3d_2\left(\alpha_1\tanh\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+\coth\left(\frac{1}{4}\alpha_1\left(tv_2+x\right)\right)+2d\right)^2}{8l_2}\end{gathered}\tag{22}$$

$$\begin{aligned} u(x,t) &= -\frac{d_1\left(d^2+2ef\right)}{l_1} - \frac{3dd_1\left(\alpha_1\sqrt{C^2+D^2}-\alpha_1 C\cosh\left(\alpha_1\left(tv_2+x\right)\right)-d\right)}{l_1\left(C\sinh\left(\alpha_1\left(tv_2+x\right)\right)+D\right)} - \\ &\quad -\frac{3d_1\left(\alpha_1\sqrt{C^2+D^2}-\alpha_1 C\cosh\left(\alpha_1\left(tv_2+x\right)\right)-d\right)^2}{2l_1\left(C\sinh\left(\alpha_1\left(tv_2+x\right)\right)+D\right)^2} \\ v(x,t) &= -\frac{6efd_2}{l_2} + \frac{3dd_2\left(\alpha_1\sqrt{C^2+D^2}-\alpha_1 C\cosh\left(\alpha_1\left(tv_2+x\right)\right)-d\right)}{l_2\left(C\sinh\left(\alpha_1\left(tv_2+x\right)\right)+d\right)} - \\ &\quad -\frac{3d_2\left(\alpha_1\sqrt{C^2+D^2}-\alpha_1 C\cosh\left(\alpha_1\left(tv_2+x\right)\right)-d\right)^2}{2l_2\left(C\sinh\left(\alpha_1\left(tv_2+x\right)\right)+D\right)^2} \end{aligned} \tag{23}$$

– *Type 2*: When $d^2 - 4ef < 0$ and $de \neq 0$ or $ef \neq 0$.

$$\begin{aligned} u(x,t) &= -\frac{d_1\left(d^2+2ef\right)}{l_1} - \frac{3dd_1e\left(\alpha_2\tan\left(\alpha_2\left(tv_2+x\right)\right)+\sec\left(\alpha_2\left(tv_2+x\right)\right)-d\right)}{l_1e} - \\ &\quad -\frac{3d_1e^2\left(\alpha_2\tan\left(\alpha_2\left(tv_2+x\right)\right)+\sec\left(\alpha_2\left(tv_2+x\right)\right)-d\right)^2}{2l_1e^2} \\ v(x,t) &= -\frac{6efd_2}{l_2} - \frac{3dd_2e(\alpha_2)\tan\left(\alpha_2\left(tv_2+x\right)\right)+\sec\left(\alpha_2\left(tv_2+x\right)\right)-d}{l_2e} - \\ &\quad -\frac{3d_2e^2\left(\alpha_2\tan\left(\alpha_2\left(tv_2+x\right)\right)+\sec\left(\alpha_2\left(tv_2+x\right)\right)-d\right)^2}{2l_2e^2} \end{aligned} \tag{24}$$

$$\begin{aligned} u(x,t) &= -\frac{d_1\left(d^2+2ef\right)}{l_1} - \frac{3dd_1\left(\alpha_2\left(\tan\left(\frac{1}{4}\alpha_2\left(tv_2+x\right)\right)-\cot\left(\frac{1}{4}\alpha_2\left(tv_2+x\right)\right)\right)-2d\right)}{2l_1} - \\ &\quad -\frac{3d_1\left(\alpha_2\left(\tan\left(\frac{1}{4}\alpha_2\left(tv_2+x\right)\right)-\cot\left(\frac{1}{4}\alpha_2\left(tv_2+x\right)\right)\right)-2d\right)^2}{8l_1} \\ v(x,t) &= -\frac{6efd_2}{l_2} - \frac{3dd_2e(\alpha_2)\tan\left(\alpha_2\left(tv_2+x\right)\right)+\sec\left(\alpha_2\left(tv_2+x\right)\right)-d)}{l_2e} \\ &\quad -\frac{3d_2e^2\left(\alpha_2\tan\left(\alpha_2\left(tv_2+x\right)\right)+\sec\left(\alpha_2\left(tv_2+x\right)\right)-d\right)^2}{2l_2e^2} \end{aligned} \tag{25}$$

$$u(x,t)=-\frac{d_1\left(d^2+2ef\right)}{l_1}-\frac{12dd_1ef\sin\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)}{l_1\left(\alpha_2\cos\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)-d\sin\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)\right)}-$$
$$-\frac{24d_1e^2f^2\sin^2\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)}{l_1\left(\alpha_2\cos\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)-d\sin\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)\right)^2}$$
$$v(x,t)=-\frac{6efd_2}{l_2}-\frac{12dd_2ef\sin\left(\frac{1}{2}\alpha_2(tv_2+x)\right)}{l_2\left(\alpha_2\cos\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)\right)-d\sin\left(\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)\right)}-$$
$$-\frac{24d_2e^2f^2\sin^2\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)}{l_2\left(\alpha_2\cos\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)-d\sin\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)\right)^2} \tag{26}$$

$$u(x,t)=-\frac{d_1\left(d^2+2ef\right)}{l_1}-\frac{12dd_1ef\sin\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)}{l_1\left(\alpha_2\left(tv_2+x\right)-d\sin\left(\alpha_2\left(tv_2+x\right)\right)+\alpha_2\cos\left(\alpha_2\left(tv_2+x\right)\right)\right)}-$$
$$-\frac{24d_1e^2f^2\sin^2\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)}{l_1\left(\alpha_2\left(tv_2+x\right)-d\sin\left(\alpha_2\left(tv_2+x\right)\right)+\alpha_2\cos\left(\alpha_2\left(tv_2+x\right)\right)\right)^2}$$
$$v(x,t)=-\frac{6efd_2}{l_2}-\frac{12dd_2ef\sin\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)}{l_2\left(\alpha_2\left(tv_2+x\right)-d\sin\left(\sqrt{4ef-d^2}\left(tv_2+x\right)\right)+\alpha_2\cos\left(\alpha_2\left(tv_2+x\right)\right)\right)}-$$
$$-\frac{24d_2e^2f^2\sin^2\left(\frac{1}{2}\alpha_2\left(tv_2+x\right)\right)}{l_2\left(\alpha_2\left(tv_2+x\right)-d\sin\left(\alpha_2\left(tv_2+x\right)\right)+\alpha_2\cos\left(\alpha_2\left(tv_2+x\right)\right)\right)^2} \tag{27}$$

- *Type 3*: When $e\neq 0$ and $d=f=0$

$$u(x,t)=-\frac{d_1\left(d^2+2ef\right)}{l_1}+\frac{6dd_1e}{l_1\left(e\left(tv_2+x\right)+h_1\right)}-\frac{6d_1e^2}{l_1\left(e\left(tv_2+x\right)+h_1\right)^2}$$
$$v(x,t)=-\frac{6efd_2}{l_2}+\frac{6dd_2e}{l_2\left(e\left(tv_2+x\right)+h_1\right)}-\frac{6d_2e^2}{l_2\left(e\left(tv_2+x\right)+h_1\right)^2} \tag{28}$$

– *Type 4*: When $f = 0$ and $de \neq 0$

$$
\begin{aligned}
u(x,t) = {} & -\frac{d_1\left(d^2+2ef\right)}{l_1}+\frac{6d_1d^2\left(\sinh\left(d\left(tv_2+x\right)\right)+\cosh\left(d\left(tv_2+x\right)\right)\right)}{l_1\left(g+\sinh\left(p\left(tv_2+x\right)\right)+\cosh\left(p\left(tv_2+x\right)\right)\right)}- \\
& -\frac{6d_1d^2\left(\sinh\left(d\left(tv_2+x\right)\right)+\cosh\left(d\left(tv_2+x\right)\right)\right)^2}{l_1\left(g+\sinh\left(p\left(tv_2+x\right)\right)+\cosh\left(p\left(tv_2+x\right)\right)\right)^2} \\
v(x,t) = {} & -\frac{6efd_2}{l_2}+\frac{6d_2d^2\left(\sinh\left(d\left(tv_2+x\right)\right)+\cosh\left(d\left(tv_2+x\right)\right)\right)}{l_2\left(g+\sinh\left(p\left(tv_2+x\right)\right)+\cosh\left(p\left(tv_2+x\right)\right)\right)}- \\
& -\frac{6d_2d^2\left(\sinh\left(d\left(tv_2+x\right)\right)+\cosh\left(d\left(tv_2+x\right)\right)\right)^2}{l_2\left(g+\sinh\left(p\left(tv_2+x\right)\right)+\cosh\left(p\left(tv_2+x\right)\right)\right)^2}
\end{aligned}
\tag{29}
$$

Graphical behavior

In this section we look over the graphical behaviour for the exact solutions of several coupled non-linear diffusion reaction equations by employing GREM. The GREM is the most successful and stable method for solving solutions to solitary waves. Solitary waves and solitons are thus self-reinforcing, envelope-preserved, high velocity waves in case of couple non-linear reaction-diffusion models. Comprehending their graphical representation entails the knowledge of how these waves interrelate and work in the said system. Solitons are peculiar forms of solitary waves that can occur in non-linear media and, in their turn, can interact with other solitons as though it were particles and, thus, can deform and collide with other solitons without changing their velocity and width. Solitary waves do not decompose as they propagate, hence are different from other waves such as dispersive or dissipative waves. When solitons meet, it is found that they appear to be undisturbed after the interaction as far as shape and speed are concerned though there may be a shift in phase. In physical systems we may see them demonstrated in optical fibers, waves on water and in plasmas, optical solitons, water wave solitons, and ion-acoustic solitons, respectively. Solitary waves are non-dispersive waves which means that the form of the wave does not affect its ability to propagate. It should however be noted that they do not have repeating forms as is the case with solitons. While solitons are only localized in the spatial co-ordinates, the amplitude of the wave vanishes as one gets further away from the core. They move at a constant velocity, they do not change their form shape No It is result of net balance of non-linear and dispersion effects. Some examples include; tsunamis because it could be considered as one large solitary wave, waves in transmission-lines, and some chemical reaction waves. The peak or the trough is more or less confined to a certain region and does not change over the entire distance as in the case of soliton or solitary wave. These signify how big the wave is or the magnitude that the wave covers. For solitons, these characteristics are invariant that means the shape and features of the solitons do not change. The wave translates at a steady rate which can be deduced from the steepness of the slant made by the wave's position in relation the time in a space-time graph. The out put of a graph involving just one soliton would involve a hump that moves along the spatial axis and can retain the form and intensity of the hump all through. It is a space-time plot representing two solitons on a course to meet, overlap at the point of intersection and still remain in their respective forms that continue to travel. An graph of a single pulse that travels in the space, it looks like a hump without spreading and changing its form. To describe numerous physical characteristics we must sketch 3-D, 2-D and their contours for the required solutions. The graphics gives us more reliable facts about the solutions behaviors. Here figs. 1 and 4 show dark soliton. Figure 2 show kink shape solitons and fig. 3 show solitary wave solitons.

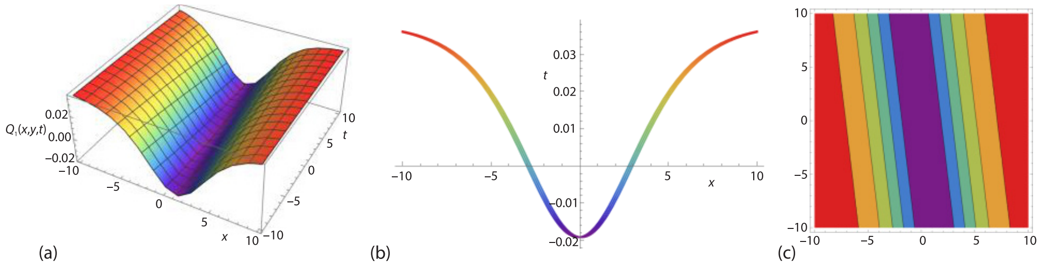

Figure 1. Above 3-D, 2-D and contour figures show the graphically behavior of eq. (7) with parameters: $d = 0.05$, $d_1 = 0.02$, $d_2 = –0.01$, $e = 0.05$, $f = –1.04$, $v_1 = 0.11$, $l_1 = –0.11$, and $v_2 = 0.12$

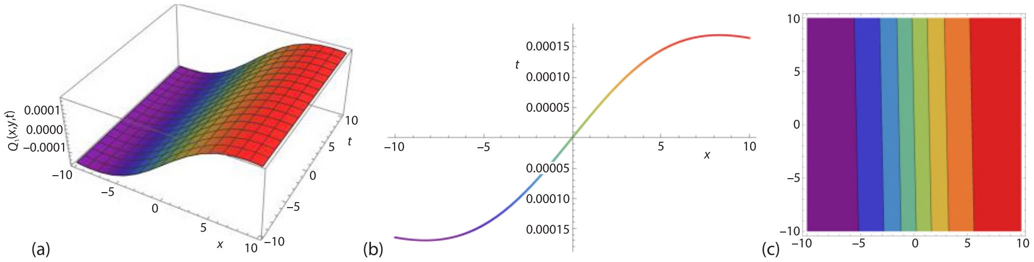

Figure 2. Above 3-D, 2-D and contour figures show the graphically behavior of eq. (8) with parameters: $d = –0.1$, $d_1 = 0.2$, $d_2 = –0.1$, $e = 0.7$, $f = –0.4$, $v_1 = 0.7$, $l_2 = 1$, and $v_2 = 2$

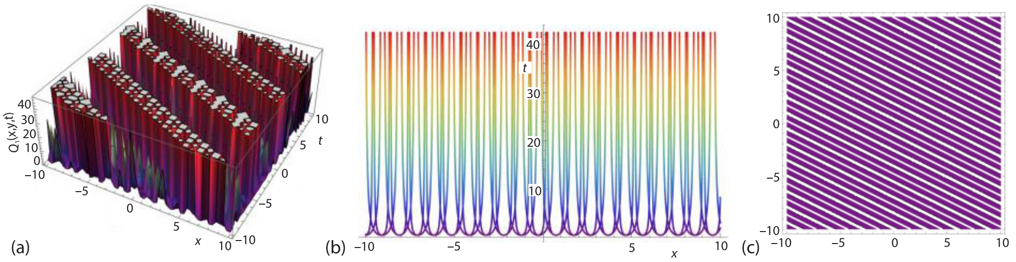

Figure 3. Above 3-D, 2-D and contour figures show the graphically behavior of eq. (11) with parameters: $d = 1.5$, $d_1 = 0.2$, $d_2 = –0.1$, $e = 1.7$, $f = 1$, $v_1 = 1$, $l_2 = 1$, and $v_2 = 2$

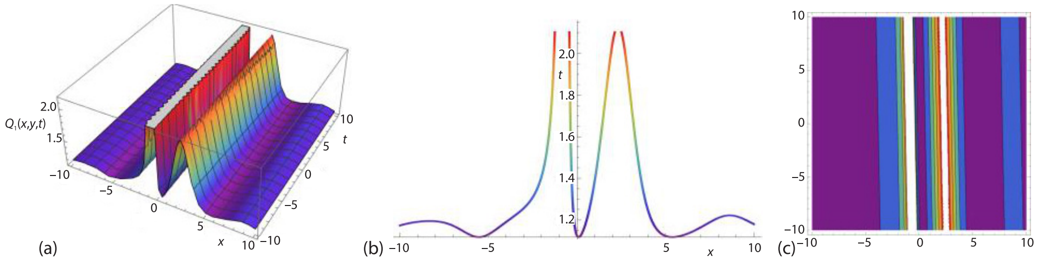

Figure 4. Above 3-D, 2-D and contour figures show the graphically behavior of eq. (14) with parameters: $d = –0.05$, $d_1 = 0.2$, $d_2 = –0.1$, $e = 0.7$, $f = 0.4$, $v_1 = 1.1$, $l_2 = 1$, and $v_2 = 0.2$

Conclusion

In this work, various structure of wave solutions like hyperbolic and trignometric were obtained for the exact solutions of a few coupled non-linear diffusion reaction equations employing GREM. The GREM approach is extra stable and effective for locating solutions to various differential and non-linear equation. The GREM technique is used for determining analytical solutions. The mathematica is operated to discover the solitary wave solutions for the needed model. Many physical significance is indicated by sketching various 3-D, 2-D and contour graph for solutions. These graphs help us to realized how the solution behave.These solutions are actually useful in the preceding discussion.

Conflict of interest

The author declares that they have no conflict of interest.

Data availability statement

Not Applicable.

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