A SIXTH-ORDER ACCURACY CONSERVATIVE LINEAR FINITE DIFFERENCE SCHEME FOR RLW EQUATION

by

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By the Taylor expansion and extrapolation combinations in the spatial direction, the second order and fourth order components of spatial truncation errors can be removed, resulting in a theoretical accuracy of sixth order. In the temporal direction, the average implicit method is employed to achieve second-order theoretical accuracy. Subsequently, a linear average implicit difference scheme for the initial boundary value problem of regularized long wave equation is constructed, which can reasonably simulate the two conservative quantities of the problem. Moreover, the convergence and stability of the scheme are also proved. Numerical examples also demonstrate the effectiveness of the proposed method.

Key words: *RLW equation, high accuracy, conservation, finite difference scheme, convergence, stability*

Introduction

The regularized long wave (RLW):

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1}$$

offers another description of the non-linear long wave equation, initially introduced by Peregrine in [1]. The RLW eq. (1) can effectively simulate nearly all applications of the well-known KdV equation and plays an important role because it can describe a large number of physical phenomena, including shallow water waves and ion waves. However, RLW eq. (1) has few analytical solutions, and its numerical investigations have also attracted a lot of attention. These numerical methods involved mainly include variational iteration method [2], allocation method [3], spectral method [4], finite element method [5], finite difference method [6, 7], and others.

In this paper, we mainly consider the initial and boundary value problems of RLW:

$$u_t + u_x + uu_x - u_{xxt} = 0, (x,t) \in (x_L, x_R) \times (0,T] \tag{2}$$

$$u(x,0) = u_0(x),\ \ x \in [x_L, x_R] \tag{3}$$

$$u(x_L,t) = u(x_R,t) = 0,\ \ t \in [0,T] \tag{4}$$

where $u_0(x)$ is a given function. Obviously, the physical boundary condition of RLW eq. (1) satisfies $u(x, t) \rightarrow 0$, $t > 0$ as $|x| \rightarrow \infty$. This initial boundary value problem possesses two conservative quantities:

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$$Q(t)=\int_{x_L}^{x_R}u(x,t)\mathrm{d}x=\int_{x_L}^{x_R}u(x,t)\mathrm{d}x=Q(0) \tag{5}$$

$$E(t)=\|u\|_{L_2}^2+\|u_x\|_{L_2}^2=\|u_0\|_{L_2}^2+\|(u_0)_x\|_{L_2}^2=E(0) \tag{6}$$

where $Q(0)$ and $E(0)$ are constants that are only related to the initial conditions.

Numerous scholars have employed various techniques to enhance accuracy. For instance, Karakoc *et al.* [8] introduced weighted parameters into a second-order scheme and adjusted the values of different weighted coefficients to optimize computational performance. The Richardson extrapolation and compact scheme are also discussed in [6, 7], which can elevate the theoretical accuracy of numerical solutions to $O(\tau^2 + h^4)$. To further advance the theoretical accuracy of numerical solutions, we apply the extrapolation combination method to eliminate the second and fourth order components of spatial truncation errors, thereby elevating the theoretical accuracy of the spatial level to sixth order. Additionally, we adopt an average linear implicit scheme in the time domain to ensure second-order theoretical accuracy. As a result, a linear difference scheme with $O(\tau^2 + h^6)$ theoretical accuracy is constructed for the initial boundary value problem (2)-(4). Notably, this scheme does not require non-linear iteration in each solution process. This approach adequately simulates the problem for the two conservative quantities (5) and (6). Furthermore, the convergence and stability of the scheme are proved. Finally, numerical experiments to verify the robustness and accuracy are carried out.

Finite difference scheme and conservative laws

For the domain $[x_L, x_R] \times [0, \mathrm{T}]$, let $h = (x_R - x_L)/J$ be the step size for the spatial grid, and τ be the step size for the temporal direction such that $x_j = x_L + jh(0 < j < J)$, $t_n = n\tau$ ($n = 0, 1, 2, ..., N$, $N = T/\tau$]. Denote $u_j^n = (x_j, t_n)$ as the exact value of $u(x, t)$ and $U_j^n \approx u\ (x_j, t_n)$ as the approximation of $u(x, t)$ at point (x_j, t_n), respectively.

Accordingly, $e_j^n = u_j^n - U_j^n$ is the error between u_j^n and U_j^n. Define:

$$(U_j^n)_x=\frac{U_{j+1}^n-U_j^n}{h},\ (U_j^n)_{\bar{x}}=\frac{U_j^n-U_{j-1}^n}{h},\ (U_j^n)_{\hat{x}}=\frac{U_{j+1}^n-U_{j-1}^n}{2h},\ (U_j^n)_{\ddot{x}}=\frac{U_{j+2}^n-U_{j-2}^n}{4h}$$

$$(U_j^n)_{\dddot{x}}=\frac{U_{j+3}^n-U_{j-3}^n}{6h},\ (U_j^n)_{\hat{t}}=\frac{U_j^{n+1}-U_j^{n-1}}{2\tau},\ \bar{U}_j^n=\frac{U_j^{n+1}+U_j^{n-1}}{2}$$

$$\left\langle U^n,V^n\right\rangle=h\sum_{j=1}^{J-1}U_j^nV_j^n,\ \left\|U^n\right\|^2=\left\langle U^n,U^n\right\rangle,\ \left\|U^n\right\|_\infty=\max_{1\le j\le J-1}\left|U_j^n\right|$$

$$Z_h^0=\left\{U=(U_j)\middle|U_{-2}=U_{-1}=U_0=U_J=U_{J+1}=U_{J+2}=0;\ j=-2,-1,0,\cdots,J,J+1,J+2\right\}$$

If the function $u(x, t)$ is smooth enough, by the Taylor expansion, we have:

$$\frac{3}{2}(u_j^n)_{\hat{x}}-\frac{3}{5}(u_j^n)_{\ddot{x}}+\frac{1}{10}(u_j^n)_{\dddot{x}}=\left(\frac{\partial u}{\partial x}\right)_j^n+O(h^6) \tag{7}$$

$$-\frac{68}{45}(u_j^n)_{x\bar{x}}+\frac{3}{5}(u_j^n)_{\hat{x}\hat{x}}-\frac{4}{45}(u_j^n)_{\hat{x}\ddot{x}}=\left(\frac{\partial^2 u}{\partial x^2}\right)_j^n+O(h^6) \tag{8}$$

Based on the aforementioned results, we propose the three-level linear average implicit finite difference scheme for the initial boundary value problem (2)-(4):

$$\begin{aligned}&(U_j^n)_{\hat{t}}-\frac{68}{45}(U_j^n)_{x\bar{x}\hat{t}}+\frac{3}{5}(U_j^n)_{\hat{x}\hat{x}\hat{t}}-\frac{4}{45}(U_j^n)_{\hat{x}\ddot{x}\hat{t}}+\frac{3}{2}(\bar{U}_j^n)_{\hat{x}}-\frac{3}{5}(\bar{U}_j^n)_{\ddot{x}}+\frac{1}{10}(\bar{U}_j^n)_{\dddot{x}}+\\&+\psi_1(U_j^n,\bar{U}_j^n)+\psi_2(U_j^n,\bar{U}_j^n)+\psi_3(U_j^n,\bar{U}_j^n)=0,\ j=1,2,\cdots,\ J-1,\ n=1,2,\cdots,N-1\end{aligned}\tag{9}$$

$$U_j^0=u_0(x_j),\ \ j=0,1,2,...,J\tag{10}$$

$$\begin{aligned}&U_j^1-\frac{68}{45}(U_j^1)_{x\bar{x}}+\frac{3}{5}(U_j^1)_{\hat{x}\hat{x}}-\frac{4}{45}(U_j^1)_{\hat{x}\ddot{x}}=u_0(x_j)-\frac{\partial^2u_0}{\partial x^2}(x_j)-\\&-\tau\frac{\partial u_0}{\partial x}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j),\ j=1,2,\cdots,\ J-1\end{aligned}\tag{11}$$

$$U^n\in Z_h^0,\ \ n=0,1,2,...N\tag{12}$$

in which

$$\psi_1(U_j^n,\bar{U}_j^n)=\frac{1}{2}\Big[U_j^n(\bar{U}_j^n)_{\hat{x}}+(U_j^n\bar{U}_j^n)_{\hat{x}}\Big],\ \psi_2(U_j^n,\bar{U}_j^n)=-\frac{1}{5}\Big[U_j^n(\bar{U}_j^n)_{\ddot{x}}+(U_j^n\bar{U}_j^n)_{\ddot{x}}\Big]$$

$$\psi_3(U_j^n,\bar{U}_j^n)=\frac{1}{30}\Big[U_j^n(\bar{U}_j^n)_{\dddot{x}}+(U_j^n\bar{U}_j^n)_{\dddot{x}}\Big]$$

Theorem 1. The scheme (9)-(12) has two discrete conservative quantities:

$$\begin{aligned}Q^n&=\frac{h}{2}\sum_{j=1}^{J-1}(U_j^{n+1}+U_j^n)+\frac{h}{4}\tau\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\hat{x}}-\frac{h}{10}\tau\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\ddot{x}}+\frac{h}{60}\tau\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\dddot{x}}=\\&=Q_n-1=\cdots=Q_0\end{aligned}\tag{13}$$

$$\begin{aligned}E^n=&\frac{1}{2}\left(\left\|U^{n+1}\right\|^2+\frac{68}{45}\left\|U_x^{n+1}\right\|^2-\frac{3}{5}\left\|U_{\hat{x}}^{n+1}\right\|^2+\left\|U^n\right\|^2+\frac{68}{45}\left\|U_x^n\right\|^2-\frac{3}{5}\left\|U_{\hat{x}}^n\right\|^2\right)+\\&+\frac{2}{45}h\sum_{j=1}^{J-1}(U_{j+1}^{n+1})_{\hat{x}}(U_j^{n+1})_{\hat{x}}+\frac{2}{45}h\sum_{j=1}^{J-1}(U_{j+1}^n)_{\hat{x}}(U_j^n)_{\hat{x}}=E^{n-1}=\cdots=E^0\end{aligned}\tag{14}$$

for $n = 1, 2,..., N-1$.

Proof. Multiplying eq. (9) with h, then summing up for j from 1 to $J-1$, by the boundary condition (12) and summation by parts [9], we have:

$$h\sum_{j=1}^{J-1}(U_j^n)_{\hat{t}}+\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(\bar{U}_j^n)_{\hat{x}}-\frac{h}{5}\sum_{j=1}^{J-1}U_j^n(\bar{U}_j^n)_{\ddot{x}}+\frac{h}{30}\sum_{j=1}^{J-1}U_j^n(\bar{U}_j^n)_{\dddot{x}}=0\tag{15}$$

Noticing:

$$h\sum_{j=1}^{J-1}U_j^n(\bar{U}_j^n)_{\hat{x}}=\frac{h}{2}\sum_{j=1}^{J-1}U_j^n(U_j^{n+1})_{\hat{x}}-\frac{h}{2}\sum_{j=1}^{J-1}U_j^{n-1}(U_j^n)_{\hat{x}}\tag{16}$$

and similar results for other terms, from the definition of Q^n, we can obtain eq. (13).

Taking the inner product of (9) with $2\bar{U}^n$, from the boundary condition (12), summation by parts [9] and the results:

$$\langle\bar{U}_{\hat{x}}^n,\bar{U}^n\rangle=0,\ \langle\bar{U}_{\ddot{x}}^n,\bar{U}^n\rangle=0,\ \langle\bar{U}_{\dddot{x}}^n,\bar{U}^n\rangle=0$$

we get:

$$\begin{aligned}&\left\|U^n\right\|_{\hat{t}}^2+\frac{68}{45}\left\|U_x^n\right\|_{\hat{t}}^2-\frac{3}{5}\left\|U_{\hat{x}}^n\right\|_{\hat{t}}^2-\frac{8}{45}\left\langle U_{\hat{x}\ddot{x}\hat{t}}^n,\bar{U}^n\right\rangle+\\&+2\left\langle\psi_1(\bar{U}^n,\bar{U}^n),\bar{U}^n\right\rangle+2\left\langle\psi_2(\bar{U}^n,\bar{U}^n),\bar{U}^n\right\rangle+2\left\langle\psi_3(\bar{U}^n,\bar{U}^n),\bar{U}^n\right\rangle=0\end{aligned}\tag{17}$$

Clearly, the results hold:

$$\left\langle U^n_{\hat{x}\ddot{x}\hat{t}},\bar{U}^n\right\rangle=-h\sum_{j=1}^{J-1}(U_j^n)_{\hat{x}\hat{t}}(\bar{U}_j^n)_{\ddot{x}}=-\frac{h}{4\tau}\sum_{j=1}^{J-1}(U_j^{n+1})_{\hat{x}}(U_{j+1}^{n+1})_{\hat{x}}+\frac{h}{4\tau}\sum_{j=1}^{J-1}(U_j^{n-1})_{\hat{x}}(U_{j+1}^{n-1})_{\hat{x}} \tag{18}$$

$$\left\langle\psi_1(U^n,\bar{U}^n),\bar{U}^n\right\rangle=\frac{h}{2}\sum_{j=1}^{J-1}\left[U_j^n(\bar{U}_j^n)_{\hat{x}}+(U_j^n\bar{U}_j^n)_{\hat{x}}\right]\bar{U}_j^n=\frac{h}{2}\sum_{j=1}^{J-1}U_j^n\bar{U}_j^n(\bar{U}_j^n)_{\hat{x}}-\frac{h}{2}\sum_{j=1}^{J-1}U_j^n\bar{U}_j^n(\bar{U}_j^n)_{\hat{x}}=0 \tag{19}$$

Similarly, we also have:

$$\left\langle\psi_2(U^n,\bar{U}^n),\bar{U}^n\right\rangle=0,\ \left\langle\psi_3(U^n,\bar{U}^n),\bar{U}^n\right\rangle=0 \tag{20}$$

Hence, substituting (18)-(20) into (17) yields (14).

Convergence and stability

The truncation error of the difference scheme (9)-(12) could be obtained:

$$\begin{aligned}r_j^n&=(u_j^n)_{\hat{t}}-\frac{68}{45}(u_j^n)_{x\bar{x}\hat{t}}+\frac{3}{5}(u_j^n)_{\hat{x}\hat{x}\hat{t}}-\frac{4}{45}(u_j^n)_{\hat{x}\ddot{x}\hat{t}}+\frac{3}{2}(\bar{u}_j^n)_{\hat{x}}-\frac{3}{5}(\bar{u}_j^n)_{\ddot{x}}+\frac{1}{10}(\bar{u}_j^n)_{\dddot{x}}+\\&+\psi_1(u_j^n,\bar{u}_j^n)+\psi_2(u_j^n,\bar{u}_j^n)+\psi_3(u_j^n,\bar{u}_j^n),\ j=1,2,\cdots,J-1,\ n=1,2,\cdots,N-1\end{aligned} \tag{21}$$

$$\begin{aligned}u_j^1-\frac{68}{45}(u_j^1)_{x\bar{x}}+\frac{3}{5}(u_j^1)_{\hat{x}\hat{x}}-\frac{4}{45}(u_j^1)_{\hat{x}\ddot{x}}&=u_0(x_j)-\frac{\partial^2u_0}{\partial x^2}(x_j)-\\-\tau\frac{\partial u_0}{\partial x}(x_j)-\tau u_0(x_j)\frac{\partial u_0}{\partial x}(x_j)+r_j^0,&\ j=1,2,\cdots,J-1\end{aligned} \tag{22}$$

By the Taylor expansion, when h, $\tau\to0$, we have $|r_j^n|=O(\tau^2+h^6)$.

Lemma 1. [10] For $\forall U\in Z_h^0$, we have:

$$\|U_{\hat{x}}\|^2\le\|U_x\|^2,\ \|U_{\ddot{x}}\|^2\le\|U_{\hat{x}}\|^2,\ \|U_{\ddot{x}}\|^2\le\|U_{\hat{x}}\|^2$$

Lemma 2. [9] Suppose that $u=\{U_j|j=0,1,2,...,J\}$ is the mesh function defined on $[x_L, x_R]$. Then, there exist two constants C_1 and C_2 such that $\|U\|_\infty\le C_1\|U_x\|+C_2\|U\|$.

Lemma 3. [9] Suppose u_0 is smooth sufficiently, and then the solution of problem (2)-(4) satisfies $\|u\|_{L_\infty}\le C$.

Theorem 2. Suppose $u_0\in H_0^1[x_L, x_R]$, and then the solution of scheme (9)-(12) satisfies:

$$\|U^n\|\le C,\ \|U_x^n\|\le C,\ \|U^n\|_\infty\le C,\ n=1,2,...,N \tag{23}$$

Proof. The proof is similar to the result in [6] and we omit the details.

Theorem 3. Suppose that $u_0\in H_0^1[x_L, x_R]$, and then the solution of scheme (9)-(12) converges to the solution of problem (2)-(4) with orde $O(\tau^2+h^6)$.

Proof. Subtracting eqs.(9) and (11) from eqs. (21)-(22) yields:

$$\begin{aligned}r_j^n&=(e_j^n)_{\hat{t}}-\frac{68}{45}(e_j^n)_{x\bar{x}\hat{t}}+\frac{3}{5}(e_j^n)_{\hat{x}\hat{x}\hat{t}}-\frac{4}{45}(e_j^n)_{\hat{x}\ddot{x}\hat{t}}+\frac{3}{2}(\bar{e}_j^n)_{\hat{x}}-\frac{3}{5}(\bar{e}_j^n)_{\ddot{x}}+\\&+\frac{1}{10}(\bar{e}_j^n)_{\dddot{x}}+\psi_1(u_j^n,\bar{u}_j^n)-\psi_1(U_j^n,\bar{U}_j^n)+\psi_2(u_j^n,\bar{u}_j^n)-\psi_2(U_j^n,\bar{U}_j^n)+\\&+\psi_3(u_j^n,\bar{u}_j^n)-\psi_3(U_j^n,\bar{U}_j^n),\ j=1,2,\cdots,\ J-1,\ n=1,2,\cdots,\ N-1\end{aligned} \tag{24}$$

$$e_j^1-\frac{68}{45}(e_j^1)_{x\bar{x}}+\frac{3}{5}(e_j^1)_{\hat{x}\hat{x}}-\frac{4}{45}(e_j^1)_{\hat{x}\ddot{x}}=r_j^0,\ j=1,2,\cdots,\ J-1 \tag{25}$$

Taking the inner product of eq. (25) with e^1, by *Lemma 1*, we have:

$$\left\|e^1\right\|^2 + \left\|e_x^1\right\|^2 \le O(\tau^2 + h^6)^2 \tag{26}$$

Then, taking the inner product of eq. (24) with $2\overline{e}^n$, summation by parts [9], and the results:

$$\langle \overline{e}_{\hat{x}}^n, \overline{e}^n \rangle = 0 \; \langle \overline{e}_{\ddot{x}}^n, \overline{e}^n \rangle = 0 \; \langle \overline{e}_{\ddot{x}}^n, \overline{e}^n \rangle = 0$$

we get:

$$\begin{aligned} \left\langle r^n, 2\overline{e} \right\rangle = {} & \left\|e^n\right\|_{\hat{t}}^2 + \frac{68}{45}\left\|e_x^n\right\|_{\hat{t}}^2 - \frac{3}{5}\left\|e_{\hat{x}}^n\right\|_{\hat{t}}^2 + \frac{2}{45\tau}h\sum_{j=1}^{J-1}(e_{j+1}^{n+1})_{\hat{x}}(e_j^{n+1})_{\hat{x}} - \frac{2}{45\tau}h\sum_{j=1}^{J-1}(e_{j+1}^{n-1})_{\hat{x}}(e_j^{n-1})_{\hat{x}} + \\ & + 2\left\langle \psi_1(u^n, \overline{u}^n) - \psi_1(U^n, \overline{U}^n), \overline{e}^n \right\rangle + 2\left\langle \psi_2(u^n, \overline{u}^n) - \psi_2(U^n, \overline{U}^n), \overline{e}^n \right\rangle + \\ & + 2\left\langle \psi_3(u^n, \overline{u}^n) - \psi_3(U^n, \overline{U}^n), \overline{e}^n \right\rangle \end{aligned} \tag{27}$$

By *Lemma 1*, *Lemma 3*, *Theorem 2*, and Cauchy-Schwarz inequality, one can get:

$$\begin{aligned} & \left\langle \psi_1(u^n, \overline{u}^n) - \psi_1(U^n, \overline{U}^n), \overline{e}^n \right\rangle = \\ & = \frac{h}{2}\sum_{j=1}^{J-1}\left[e_j^n(\overline{u}_j^n)_{\hat{x}} + U_j^n(\overline{e}_j^n)_{\hat{x}}\right]\overline{e}_j^n - \frac{h}{2}\sum_{j=1}^{J-1}\left[e_j^n\overline{u}_j^n + U_j^n\overline{e}_j^n\right](\overline{e}_j^n)_{\hat{x}} \le \\ & \le C\left(\left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2\right) \end{aligned} \tag{28}$$

and

$$\langle r^n, 2\overline{e}^n \rangle \le \left\|r^n\right\|^2 + \left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2 \tag{29}$$

Similar results can also obtained for ψ_2 and ψ_3. Consequently, we have:

$$\begin{aligned} & \left\|e^n\right\|_{\hat{t}}^2 + \frac{68}{45}\left\|e_x^n\right\|_{\hat{t}}^2 - \frac{3}{5}\left\|e_{\hat{x}}^n\right\|_{\hat{t}}^2 + \frac{2}{45\tau}h\sum_{j=1}^{J-1}(e_{j+1}^{n+1})_{\hat{x}}(e_j^{n+1})_{\hat{x}} - \frac{2}{45\tau}h\sum_{j=1}^{J-1}(e_{j+1}^{n-1})_{\hat{x}}(e_j^{n-1})_{\hat{x}} \le \\ & \le \left\|r^n\right\|^2 + C\left(\left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2\right) \end{aligned} \tag{30}$$

Letting:

$$\begin{aligned} B^n = {} & \left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \frac{68}{45}\left\|e_x^{n+1}\right\|^2 + \frac{68}{45}\left\|e_x^n\right\|^2 - \frac{3}{5}\left\|e_{\hat{x}}^{n+1}\right\|^2 - \frac{3}{5}\left\|e_{\hat{x}}^n\right\|^2 + \\ & + \frac{4}{45}h\sum_{j=1}^{J-1}(e_{j+1}^{n+1})_{\hat{x}}(e_j^{n+1})_{\hat{x}} + \frac{4}{45}h\sum_{j=1}^{J-1}(e_{j+1}^n)_{\hat{x}}(e_j^n)_{\hat{x}} \end{aligned}$$

and summing up eq. (30) for j from 1 to n, we have

$$B^n \le B^0 + 2\tau\sum_{l=1}^{n}\left\|r^l\right\|^2 + C\tau\sum_{l=0}^{n+1}(\left\|e^l\right\|^2 + \left\|e_x^l\right\|^2) \tag{31}$$

Therefore, from *Lemma 1*, (31) can be re-written:

$$\left\|e^{n+1}\right\|^2 + \left\|e^n\right\|^2 + \frac{37}{45}\left\|e_x^{n+1}\right\|^2 + \frac{37}{45}\left\|e_x^n\right\|^2 \le B^n \le O(\tau^2 + h^6)^2 + C\tau\sum_{l=0}^{n+1}(\left\|e^l\right\|^2 + \left\|e_x^l\right\|^2)$$

By the discrete Gronwall inequality [9], we get:

$$\left\|e^n\right\| \le O(\tau^2 + h^6), \; \left\|e_x^n\right\| \le O(\tau^2 + h^6)$$

Finally, an application of *Lemma 2* leads to:

$$\left\| e^n \right\|_\infty \le O(\tau^2 + h^6)$$

Numerical experiments

The single solitary-wave solution of RLW eq. (1) [10] is taken:

$$u(x,t) = \sec h^2 \left(\frac{1}{4} x - 3t \right)$$

and the corresponding initial function of problem (1)-(3) is given by $u(x, 0) = \sec h^2(x/4)$.Take $x_L = -30$, $x_R = 80$, and $T = 32$. To test the theoretical accuracy of the numerical solutions of the difference schemes (9)-(12) with different norms, define:

$$\text{order} l_2 = \log_2 \left(\frac{\left\| e^n (h,\tau) \right\|}{\left\| e^{8n} (h/2, \tau/8) \right\|} \right)$$

$$\text{order} l_\infty = \log_2 \left(\frac{\left\| e^n (h,\tau) \right\|_\infty}{\left\| e^{8n} (h/2, \tau/8) \right\|_\infty} \right)$$

For different τ and h, the accuracy verification of numerical solutions of the difference scheme (9)-(12) are listed in tab. 1. Meanwhile, numerical simulations for conservative quantities eqs. (5) and (6) are presented in tab. 2.

Table 1. The numerical verification of theoretical accuracy $O(\tau^2 + h^6)$

	orderl_2			orderl_∞		
	$\tau = 0.4$, $h = 0.1$	$\tau = 0.05$, $h = 0.05$	$\tau = 000625$, $h = 0.025$	$\tau = 0.4$, $h = 0.1$	$\tau = 0.025$, $h = 0.05$	$\tau = 000625$, $h = 0.025$
$t = 8$	–	5.715	5.964	–	5.687	5.991
$t = 16$	–	5.667	5.972	–	5.627	5.992
$t = 24$	–	5.633	5.977	–	5.579	5.992
$t = 32$	–	5.607	5.981	–	5.541	5.991

Table 2. Conservative quantities eqs. (5) and (6) for variousτ and h

	Q^n			E^n		
	$\tau = 0.4$, $h = 0.1$	$\tau = 0.05$, $h = 0.05$	$\tau = 0.0625$, $h = 0.025$	$\tau = 0.4$, $h = 0.1$	$\tau = 0.05$, $h = 0.05$	$\tau = 0.00625$, $h = 0.025$
$t = 0$	8.036087	8.000589	8.000006	5.599985	5.600000	5.600000
$t = 8$	8.036091	8.000597	8.000054	5.599985	5.600000	5.600000
$t = 16$	8.036092	8.000602	8.000105	5.599985	5.600000	5.600000
$t = 24$	8.036092	8.000148	8.000148	5.599985	5.600000	5.600000
$t = 32$	8.036092	8.000184	8.000184	5.599985	5.600000	5.600000

Conclusion

As demonstrated in the tables, the difference scheme (9)-(12) proposed in this paper for problem (2)-(4) exhibits second-order accuracy in time and sixth-order accuracy in space. It also accurately simulates the conservation properties eqs. (5) and (6). Furthermore, the scheme is linear, eliminating the need for non-linear iterations in the numerical solution process and resulting in relatively less computational time.

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Nomenclature

t – time, [second]
u – the fluid velocity, [ms^{-1}]
x – co-ordinates, [m]

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