THE LIE GROUP ANALYSIS METHOD FOR HEAT TRANSFER IN STEADY BOUNDARY-LAYER FLOW FIELD OF NANOFLUID

by

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In order to reveal the mechanism of the abnormal movement (Brownian motion enhances thermal scattering) of nanoparticles on the fluid enhanced heat transfer, the two-phase model was used to study the abnormal convection and diffusion of viscous nanofluids in the flat boundary-layer of porous medium. Firstly, for the 2-D steady boundary-layer stagnation point flow of incompressible Newtonian-nanofluids, the non-linear governing equations of the flow field and temperature field of nanofluids are established from the Oberbeck-Boussinesq approximate equations. Secondly, the modern Lie group analysis method is introduced, we give the Lie symmetry determining equation of the flow field PDE and the characteristics of the solutions. Further, using the relationship between the Lie symmetries and the conserved quantities, the conservation vector form of the flow field and the group invariant solution are derived in detail, and the reduced order model of the nanofluid flat boundary-layer is obtained. Finally, the correctness of the analytical results obtained by the Lie group method was verified for different values of the flow parameter Prandtl. Research has shown that the Lie group method can be used to analytically solve the velocity and temperature distribution functions of abnormal motion of nanoparticles. The fluid temperature increases with the increase of the volume fraction parameter of nanoparticles, but decreases with the increase of the Prandtl value of the base fluid, and decreases with the increase of the plate stretching speed. The Lie group analysis method in this paper provides reference value for numerical simulation solutions of various heat and mass transfer in nanofluids.

Key words: *nanofluids, plate boundary-layer, lie group analysis, conservative vector, invariant solution*

Introduction

Nanofluids [1, 2] are colloidal suspensions containing nanoscale particles in liquids, which can significantly improve thermal conductivity compared to pure base liquids. The physical properties of nanofluids are highly anomalous due to the complex force interactions between nanoparticles, and between particles and base liquids. Numerous studies have shown [3, 4] the high thermal conductivity of nanofluids is closely related to the anomalous diffusion motion of nanoparticles, which is a comprehensive result of Brownian motion of particles,

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agglomeration effect and surface layer effect of particles [5]. This is of great significance for improving the utilization efficiency of chemical energy. The latest technological applications for heat and mass transfer of nanoparticles is ultrasound regulates the movement of nanoparticles, for example, Hedeshi [6] investigated the simultaneous effects of using nanofluid and ultrasonic vibrations on the heat transfer enhancement of a double-pipe heat exchanger, Amin *et al*. [7] investigated an innovative active method based on the ultrasonication technology to improve heat transfer in low thermal efficiency of indirect water bath heaters. It is worth noting that the mechanical models used in the aforementioned literatures for heat and mass transfer in nanofluids are all boundary-layer equations. The mass and momentum exchange mechanism of nanofluids in boundary-layer flow on porous extension planes is an important scientific topic, as their temperature and density distribution characteristics can directly affect some polymer forming and metallurgical processes [8]. From the perspective of mathematical deduction mechanism, the problem of anomalous heat transfer in nanofluids can be described by non-linear Navier-Stokes dynamic equations. Therefore, finding approximate or exact solutions to the anomalous convection diffusion equation in the boundary-layer of nanofluid flat plates is a very meaningful fundamental theoretical work.

At present, domestic and foreign scholars have done a lot of work on the numerical simulation and experimental aspects of heat and mass transfer of nanofluids under various mechanical conditions. For example, Xu and Huang [9] used fourth-order Runge-Kutta numerical calculations for the boundary-layer problem of nanofluids containing rotating microorganisms. Aouinet *et al*. [10] numerically simulated the turbulent boundary-layer of three different nanofluids on a flat plate using a CFD program. Kalpana *et al*. [11] conducted a numerical simulation study on the unsteady MHD nanofluid boundary-layer flow under the combined action of Brownian motion and thermophoresis. Dong *et al*. [12] experimentally measured the thermal conductivity data of SiO_2 nanofluids with different mass concentrations and temperatures. Darvishi *et al*. [13] used the mathematical analysis on heat equation and generating graphs for finding the parameters important to the heat transfer in the straight fins. Although numerical simulation can save time, the accuracy of numerical difference algorithms is generally limited by step size and introduces a certain amount of dissipation error, making it difficult to display the quantitative relationship between mechanical response and structural parameters globally. Although experimental methods are accurate and reliable, they often consume time and effort. Imagine if we could reduce the order or approximate analytical solution of the boundary-layer equation for nanofluids, which would greatly facilitate the understanding of the structure and properties of the original system. The Lie group analysis method for differential equations was originally proposed by Sophius Lie in the 1950's. It has been proven that the Lie group method is a very universal and effective algorithm for solving exact solutions to non-linear PDE. The advantages of using the Lie group analysis method compared to other numerical or experimental approaches in studying nanofluid mechanics problems are that Lie symmetry group transformations can convert the boundary-layer equations into non-linear ODE. The coupled non-linear PDE are transformed to ODE by utilizing an one parameter Lie-group analysis and then the spectral local linearization method is applied to get the numerical values for the surface friction and Nusselt number along with the entropy and Bejan number profiles, which makes this problem meaningful for the research in fluid dynamics. Therefore, applying the Lie group algorithm to the anomalous diffusion heat transfer in the boundary-layer of nanofluid flat plates has feasibility and certain advantages.

The use of Lie group analysis method to study nanofluid mechanics problems has achieved some research results. For example, in [14], a single parameter Lie group was applied to solve the problem of the influence of magnetic field on free convection of nanofluids on semi-infinite flat plates, and a similar reduction was given. Kandasamy and Haimin [15] used Lie group transformation to study the temperature dependent fluid viscosity and the effect of thermal particle deposition on free heat and mass transfer under different flow conditions. Pan [16] used the Lie group analysis method to obtain the conservation law of the flow equation and the variation law of velocity and temperature fields for the boundary-layer problem of fractional viscoelastic fluids. Rosmila *et al.* [17] used Lie group variation to study the natural convection of MHD in nanofluids flowing through a thermally stratified linear porous extension plane, and obtained the key factors affecting the high thermal conductivity of nanofluids. The research and application of modern mathematical tools such as Lie groups not only enable creative solutions to complex and profound problems in fluid dynamics, but also promote the further development of Lie group theory. Overall, the Lie group analysis method currently used in the study of heat and mass transfer in nanofluids is not systematic and relatively scattered, and there is no standardized program process. Moreover, there is very little research specifically focused on the problem of anomalous convection and diffusion in the boundary-layer of nanofluid flat plates. The purpose of this article is to provide a general algorithm for Lie group analysis in nanofluid dynamics, which involves transforming and deducing the Lie symmetry generating element solution characteristics, conservation vector expression forms, group invariant solution construction, and the influence of different nanofluid parameters on flow and heat transfer characteristics when nanofluid flows through an extended surface.

The novelty of the work:

- From a physical perspective, boundary-layer equations are particularly interesting as they allow for a large number of invariant solutions, namely basic analytical solutions. By using Lie symmetry group transformation, the partial differential control equations for mass, momentum, thermal energy, and particle concentration conservation in nanofluid mechanics are transformed into ODE, this method helps in reducing the challenges encountered while solving equations due to the PDE non-linear nature.
- The control equations of nanofluid boundary-layer comprehensively considers the effects of Brown motion and thermal swimming. It is found that the volume fraction of nanoparticles is a key parameter for studying the effects of nanoparticle flow field and temperature distribution the nanofluid boundary-layer. It can also be seen that numerical solutions can only exist when the boundary and initial conditions meet certain conditions, however, the Lie group analysis method is a unique and rigorous mathematical method for finding all symmetries and similarity solutions for boundary-layer flow field of nanofluid, without the need for special assumptions.

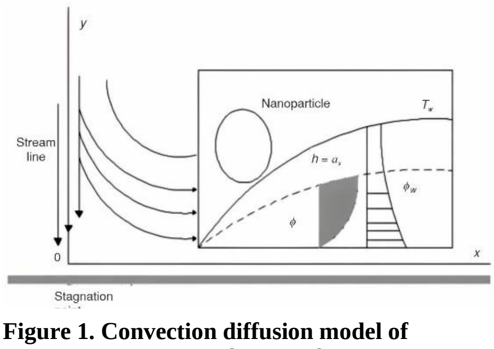

Figure 1. Convection diffusion model of plate boundary-layer for nanofluid

Convection diffusion of nanofluid

Considering the incompressible Newtonian viscous fluid, the steady 2-D laminar flow is carried out on the plane $y = 0$. The flow is limited $y > 0$, it is shown in fig. 1.

Two forces of equal magnitude and opposite directions are applied along the x-axis to make the wall extend while keeping the position of the origin unchanged [18], the stretching or shrinking speed of plate is a, the temperature is constant, T_w, and the volume fraction of nanoparticles is ϕ_w on the wall of plate. The fluid temperature below the plate is the same as that of the wall. When y tends to infinity, the temperature is constant T_∞ and the volume fraction of nanoparticles is ϕ_∞ and the outflow velocity of boundary-layer h_∞ is 0. Furthermore, it is assumed that the conventional fluid and the suspended nanoparticles are in thermal equilibrium, and there is no thermal expansion and relative sliding between them. If the heat source and fluid viscosity dissipation are not considered, the control equations of nanofluid satisfy the previous conditions is obtained by boundary-layer theory:

$$\frac{\partial h}{\partial x}+\frac{\partial v}{\partial y}=0,\ h\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}=\frac{\mu_{\mathrm{nf}}}{\rho_{\mathrm{nf}}}\frac{\partial^2 h}{\partial y^2},\ h\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{\mathrm{nf}}\frac{\partial^2 T}{\partial y^2}$$
$$h\frac{\partial \varphi}{\partial x}+v\frac{\partial \varphi}{\partial y}=D_B\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+\frac{D_T}{T_\infty}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial x^2}\right) \tag{1}$$

where the first to fourth equations in eq. (1) represent the conservation of mass, momentum, thermal energy, and nanoparticles volume, respectively, x, y are the co-ordinates of plane, h, v are the velocity components in x-, y-direction, T – the local temperature of fluid, ϕ – the volume fraction of nanoparticles, ρ_{nf} – its effective density, μ_{nf} – its effective dynamic viscosity, $(\rho c_p)_{\mathrm{nf}}$ – its effective heat capacity, α_{nf} – its effective thermal diffusivity, D_B – the diffusion coefficient of brown motion (the proportionality coefficient of the mean square displacement of the particle motion with respect to time), and D_T – the diffusion coefficient of thermophoresis (the proportional coefficient of the thermophoresis velocity to temperature gradient). For the previous parameters, please refer to [19]:

$$\alpha_{nf}=\frac{k_{\mathrm{nf}}}{(\rho c_p)_{\mathrm{nf}}},\ \mu_{\mathrm{nf}}=\frac{\mu_f}{(1-\varphi)^{2.5}}$$
$$\rho_{\mathrm{nf}}=(1-\varphi)\rho_f+\varphi\rho_s,\ (\rho c_p)_{\mathrm{nf}}=(1-\varphi)(\rho c_p)_f+\varphi(\rho c_p)_s \tag{2}$$
$$k_{\mathrm{nf}}=k_f\frac{k_s+2k_f-2\varphi(k_f-k_s)}{k_s+2k_f+2\varphi(k_f-k_s)},\ D_B=\frac{K_BT}{6\pi\eta R},\ D_T=\frac{\alpha_{\mathrm{nf}}}{\rho c_p}$$

where f, s mean the base fluid and nanoparticles, k_{nf} – the effective heat exchange rate, ρ, c_p are the reference standard density and specific heat capacity, K_B is Boltzmann constant, R is the radius of nanoparticles, and η – the liquid viscosity.

The boundary conditions are as follows:

$$h=H=ax,\ v=0,\ T=T_w,\ \varphi=\varphi_w,\quad y=0$$
$$h=0,\ T=T_\infty,\ \varphi=\varphi_\infty,\qquad y=\infty \tag{3}$$

Setting the stream function is $\psi(x,y)$, $h=\partial\psi/\partial y$, $v=-\partial\psi/\partial x$, so the dynamic boundary-layer equations of flow field and temperature field are as:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\frac{\mu_{\mathrm{nf}}}{\rho_{\mathrm{nf}}}\frac{\partial^3\psi}{\partial y^3}$$
$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}=\alpha_{\mathrm{nf}}\frac{\partial^2 T}{\partial y^2} \tag{4}$$

Lie symmetries of the plate boundary-layer equation for nanofluid

The independent variable is $\boldsymbol{x}=(x,y)$, the dependent variable is $\boldsymbol{u}=(\psi,T)$, the total differential operator, first and second order partial derivatives are:

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} \tag{5}$$

The infinitesimal transformation group with a single parameter:

$$\begin{aligned} x^* &= x + \varepsilon\xi_1(x,y,\psi,T),\ z^* = z + \varepsilon\xi_2(x,y,\psi,T) \\ \psi^* &= \psi + \varepsilon\eta_1(x,y,\psi,T),\ T^* = T + \varepsilon\eta_2(x,y,\psi,T) \end{aligned} \tag{6}$$

where ε is a infinitesimal parameter and ξ_i, η_i are infinitesimal generator functions.

According to the extension theory of Lie group, the first, second, and third order extension vector fields of (6) are:

$$\begin{aligned} &X = \xi_i(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x_i} + \eta_r(\mathbf{x},\mathbf{u})\frac{\partial}{\partial u^r},\ X^{(1)} = X + \zeta_i^{(1)w}(\mathbf{x},\mathbf{u},\partial\mathbf{u})\frac{\partial}{\partial u_i^w} \\ &X^{(2)} = X^{(1)} + \zeta_{i_1 i_2}^{(2)v}(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^2\mathbf{u})\frac{\partial}{\partial u_{i_1 i_2}^v},\ X^{(3)} = X^{(2)} + \zeta_{i_1 i_2 i_3}^{(3)q}(\mathbf{x},\mathbf{u},\partial\mathbf{u},\partial^3\mathbf{u})\frac{\partial}{\partial u_{i_1 i_2 i_3}^q} \\ &\zeta_i^{(1)w} = D_i\eta^w - (D_i\xi_j)u_j^w,\ \zeta_{i_1 i_2}^{(2)v} = D_{i_2}\zeta_{i_1}^{(1)v} - (D_{i_2}\xi_j)u_{i_1 j}^v \\ &\zeta_{i_1 i_2 i_3}^{(3)q} = D_{i_3}\zeta_{i_1 i_2}^{(2)q} - (D_{i_3}\xi_j)u_{i_1 i_2, j}^q,\ (i,j,i_1,i_2,i_3,r,w,v,q = 1,2) \end{aligned} \tag{7}$$

where $\xi^{(1)}$ is i^{th} order infinitesimal generator function.

The invariance of eq. (4) under the transformation (6) can be expressed as:

$$X^{(3)}\left[\frac{\mu_{\text{nf}}}{\rho_{\text{nf}}}\frac{\partial^3\psi}{\partial y^3} - \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right] = 0,\ X^{(2)}\left[\alpha_{\text{nf}}\frac{\partial^2 T}{\partial y^2} - \frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right] = 0 \tag{8}$$

The eq. (8) is expanded to solve the high-order linear PDE of ξ_1, ξ_2, η_1, η_2, and the coefficient expression is:

$$\xi_1 = c_1\frac{x}{\sqrt{\frac{\nu_f}{a}}} + c_2,\ \xi_2 = g\left(\frac{x}{\sqrt{\frac{\nu_f}{a}}}\right),\ \eta_1 = c_3\psi + c_4,\ \eta_2 = c_5\frac{T - T_\infty}{T_\omega - T_\infty} \tag{9}$$

where g is an arbitrary function, $c_1 \sim c_5$ are arbitrary constants, $\nu_f = \mu_f/\rho_f$ is dynamic viscosity coefficient of base fluid. All generators X of eq. (8) are constructed the commutator operation $[\bullet]$ (Lie bracket) in partial differential operator space, the basement of the corresponding four-dimensional Lie algebra structure is:

$$X_1 = \frac{x}{\sqrt{\frac{\nu_f}{a}}}\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + \nu_f\psi\frac{\partial}{\partial\psi} + \frac{T - T_\infty}{T_\omega - T_\infty}\frac{\partial}{\partial T},\ X_2 = \frac{\partial}{\partial x} + g\frac{\partial}{\partial y},\ X_3 = g\frac{\partial}{\partial y} + \frac{\partial}{\partial\psi} \tag{10}$$

The Lie algebra represents the tangent space near the unit element of a Lie group, which is a linear vector space. The X_1, X_2, X_3 in eq. (10) are independent of each other. The X_1 corresponds to scaling group of transformation, while the X_2 and X_3 are translation groups of transformation.

Conservation law of the plate boundary-layer equation for nanofluid

For the solution $\boldsymbol{u} = \boldsymbol{u}(x)$ of eq. (4), if the vector $C(C^1(x,\boldsymbol{u},\partial\boldsymbol{u},\partial^2\boldsymbol{u}),\ C^2(x,\boldsymbol{u},\partial\boldsymbol{u},\partial^2\boldsymbol{u}))$ satisfies:

$$div\mathbf{C} = D_x(C^1) + D_y(C^2) = 0 \tag{11}$$

So $\boldsymbol{C}$ is a conservation vector of eq. (4).

According to the theorem 1 in [20], if the allowed Lie symmetric transformation generator vector of PDE:

$$F^m(\boldsymbol{x},\boldsymbol{u},\partial\boldsymbol{u},\partial^2\boldsymbol{u},\partial^3\boldsymbol{u}) = 0,\ X = \xi_i(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial x_i} + \eta_i(\boldsymbol{x},\boldsymbol{u})\frac{\partial}{\partial u^i},\ (i=1,2)$$

then the conservation vector $\boldsymbol{C}$ of PDE must satisfies:

$$C^i = \xi_i L + W^\alpha\left[\frac{\partial L}{\partial u_i^\alpha} - D_j\left(\frac{\partial L}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right)\right] + D_j(W^\alpha)\left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right)] + D_j D_k(W^\alpha)\left(\frac{\partial L}{\partial u_{ijk}^\alpha}\right)\right] \tag{12}$$
$$W^\alpha = \eta_\alpha - \xi_j u_j^\alpha$$

where L is Lagrange form function.

It is noted that the plate boundary-layer eq. (4) for nanofluid satisfies the self-adjoint condition, so the Lagrange form function of flow field can be taken as:

$$L = c_6\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \frac{\mu_{nf}}{\rho_{nf}}\frac{\partial^3\psi}{\partial y^3}\right] \tag{13}$$

where c_6 □0 is a constant.

By substituting the infinitesimal Lie symmetry generator in (10) and (13) into the eq. (12), can obtain two sets of conservative vectors (nontrivial) of eq. (4):

$$CL_1: C^1 = \frac{1}{2}\psi_y^2 - \frac{3}{2}\psi\psi_{yy},\ C^2 = -\frac{\mu_{nf}}{\rho_{nf}}\psi_y + \frac{3}{2}\psi\psi_{xy} - \psi_x\psi_y$$
$$CL_2: C^1 = y\psi_y\psi_{yy} - \frac{1}{2}\psi_y^2, C^2 = \frac{\mu_{\text{nf}}}{\rho_{\text{nf}}}(\psi_y + y\psi_{yy}) + \psi_x\psi_y + y(\psi_x\psi_{yy} - \psi_y\psi_{xy}) - \frac{\sqrt{\frac{\nu_f}{a}}}{2}\psi_x\psi_{xy} \tag{14}$$

According to the conservation law of Newtonian fluid wall jet in [21], the nanofluid principle includes mass conservation, momentum conservation, thermal energy conservation and nanoparticle conservation. The first set in (14) is the momentum conservation equation, so the physical meaning is momentum density conservation, and the conserved quantity of the second set in (14) does not conform to any one of the control equations, it is only a mathematical expression.

Invariant solutions of the plate boundary-layer equation for nanofluid

In order to obtain new exact solutions from the existing solutions of equations, need group invariant solution of equations. If the $\boldsymbol{u} = J(\boldsymbol{x})$ is a invariant solution of PDE:

$$F^m(\boldsymbol{x},\boldsymbol{u},\partial\boldsymbol{u},\partial^2\boldsymbol{u},\partial^3\boldsymbol{u}) = 0$$

and the solution comes from the Lie symmetric transformation generator vector of PDE, if and only if $X(J) = 0$ are established. The group invariant solution combined with the separation of variables can reduce the number of independent variables, and then it can be transformed into ODE, which provides a way to get the exact solution of PDE [22].

Taking a special infinitesimal generator, its invariant solution is:

$$\frac{x}{\sqrt{\frac{\nu_f}{a}}}\frac{\partial J}{\partial x}+\upsilon_f\psi\frac{\partial J}{\partial \psi}+\frac{T-T_\infty}{T_w-T_\infty}\frac{\partial J}{\partial T}=0 \tag{15}$$

The corresponding characteristic equation is:

$$\frac{\mathrm{d}x}{\frac{x}{\sqrt{\frac{\nu_f}{a}}}}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}\psi}{\nu_f\psi}=\frac{\mathrm{d}T}{\frac{T-T_\infty}{T_w-T_\infty}} \tag{16}$$

So the invariant is $\lambda = y(\nu_f/a)^{1/2}$ and the inform of variant solution is:

$$\psi=\sqrt{a\nu_f}\,xf(\lambda),\ T=T_\infty+(T_w-T_\infty)x\theta(\lambda) \tag{17}$$

Substituting (17) into (4), have:

$$\phi_1 f'''(\lambda)+f(\lambda)f''(\lambda)-f'^2(\lambda)=0,\ \phi_2\theta''(\lambda)+f(\lambda)\theta'(\lambda)-\theta(\lambda)f'(\lambda)=0 \tag{18}$$

Here

$$\phi_1=\frac{1}{(1-\varphi)^{2.5}\left(1-\varphi+\frac{\varphi\rho_s}{\rho_f}\right)},\ \phi_2=\frac{1}{\mathrm{Pr}}\frac{\frac{k_{\mathrm{nf}}}{k_f}}{\left[1-\varphi+\frac{\varphi(\rho c_p)_s}{(\rho c_p)_f}\right]},\ \mathrm{Pr}=\frac{(\mu c_p)_f}{k_f}$$

μ is reference standard viscosity.

Then the invariant solution of velocity field equation is:

$$f(\lambda)=\phi_1(1-\mathrm{e}^{-\lambda}) \tag{19}$$

The temperature field equation is a second order ODE with variable coefficients, which can be expressed by the Kummer function, M_k, and can also be solved iteratively by numerical difference algorithm. The results show that the volume fraction of nanoparticles is a key parameter to study the effect of flow field and temperature distribution of nanofluid.

The local wall friction coefficient, C_f, and local Nusselt number, Nu_x, have important applications in practical problems, which are:

$$C_f(\mathrm{Re}_x)^{0.5}=\frac{1}{(1-\varphi)^{2.5}}f''(0),\ \mathrm{Nu}_x(\mathrm{Re}_x)^{-0.5}=\frac{k_{\mathrm{nf}}}{k_f}\theta'(0) \tag{20}$$

where $\mathrm{Re}_x = Hx/\nu_f$ is the local Reyonlds number.

Now, selecting Cu nanoparticles in numerical calculations in this paper. The values of density, size, heat conductivity, and heat capacity of the nanoparticles can be obtained by looking up tab. 1.

Table 1. Thermophysical properties of fluid and nanoparticles

Nanoparticles is Cu, fluid is water	Density	Size	Heat conductivity	Heat capacity	Particle volume fraction	D_B	D_T
	8933 [kgm^{-3}]	20 nm	401 [Wm^{-1}K^{-1}]	385 [Jkg^{-1}K^{-1}]	0.05	2×10^{-18} [m^2 s^{-1}]	1.67×10^{5} [K^{-1}]

In order to illustrate the correctness, the control parameters are taken to satisfy the boundary conditions. The comparison with the available solutions in reference is shown in tab. 2. The comparison results show there is a good consistency for each Prandtl value. Therefore, the results of this paper are very accurate.

Table 2. Comparisonof results for Nusselt $-\theta'(0)$ with the reference [23]

Pr	The paper $-\theta'(0)$	[23]
0.72	0.80867991	0.8086
1.00	1.00000000	1.0000
3.00	1.92368891	1.9237
7.00	3.72077927	3.7207

When the parameters meet the boundary conditions, the step size $\Delta\lambda = 0.1$ is taken, and the fourth-order Runge-Kutta algorithm is used to solve the ODE (18). The temperature field curves under different Prandtl number are given in fig. 2(a), and the results of Mukhopadhyay and layek [24] are given in fig. 2(b), we have found that they agree well. It can be seen that the fluid temperature decreases with the increase of the Prandtl value, which is caused by the thinning of thermal boundary-layer. This characteristic shows that the smaller Prandtl number of fluids is more sensitive to thermal radiation.

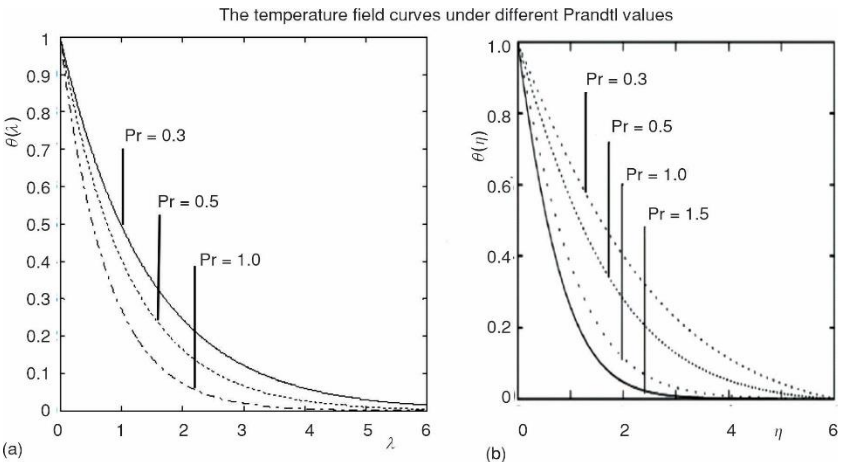

Figure 2. The temperature field curves under different Prandtl number values; (a) results of this paper and (b) results of [24]

Conclusion

The Lie group analysis method is a type of analytical and deductive algorithm that starts from the dynamic differential equation itself. Its core is that the invariance of the differential equation under group transformation can construct the first integral and group invariant solution of the original equation, which provides a good idea for the reduction, reduction, and exact general solution of the original equation. The difficulty of this algorithm lies in the effective solution of Lie symmetric generator functions. In response to the problem of anomalous convection and diffusion in the boundary-layer of viscous nanofluids in porous media, this paper uses modern Lie group method to solve the Lie symmetry, conservation vector, and group invariant solution of the nanofluid boundary-layer equation system. The reduced order model of the original PDE system is obtained, and the heat transfer index characteristics in the boundary-layer of the nanofluid plate are analytically calculated using the reduced order model.

The research conclusions mainly include:

- The Lie symmetric generator vector allowed by the anomalous convection diffusion equation system of the nanofluid flat plate boundary-layer is a 4-D partial differential operator, and its Lie algebraic space has a 3-D structure.
- There exists a conservation vector in the anomalous convection diffusion equation system of the nanofluid plate boundary-layer, which reflects the conservation of momentum density of the nanofluid.
- The Lie group invariant solution of the anomalous convection diffusion equation system in the boundary-layer of a nanofluid flat plate can reduce the original equation to a third-order ODE system for velocity field and second-order ODE system for temperature field.
- The temperature in the boundary-layer of nanofluid flat plates decreases with increasing plate stretching speed and Prandtl number. The Lie group method in this article can be further extended to boundary-layer problems of heat and mass transfer in non-stationary and non-Newtonian nanofluids.

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