IDENTIFYING OF UNKNOWN SOURCE TERM FOR THE RAYLEIGH-STOKES PROBLEM

by

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In this paper, we would like to briefly introduce some applications of fractional derivatives in the fields of heat and fluid-flows. However, our main focus is on study an inverse source problem for the Rayleigh-Stokes problem. The problem is severely ill-posed. We verify the ill-posedness of Problem 1, applying the modified Lavrentiev to construct a regularization from the exact data. After that, we established the convergent rate between the exact solution and its approximation. Furthermore, we have the estimate in L^q space.

Key words: *unknown source, Rayler-Stokes problem, regularization method, inverse problem*

Introduction

In the last few decades, the study of fractional models has received a lot of attention, in which the inverse problems and the problem of determining the error source function play an important role in engineering applications, mathematical finance, physics, image processing mechanisms, and continuous media, [1-13]. Duan *et al*. [14] and Trasov [15], they had plan to structure the fractal heat transfer equations in fractal media at low and high excess temperatures. In this reference, their main aim is to propose the linear and non-linear heat transfer equations from the local fractional calculus point of view and to present the linear and non-linear oscillator equations arising in fractal heat transfer. Khan *et al*. [16], authors studied dealt with an exact solution for the MHD flow of a generalized Oldroyd-B fluid in a circular pipe. For the description of such a fluid, the fractional calculus approach has been used throughout the analysis Based on modified Darcy's law for generalized Oldroyd-B fluid. The model Rayleigh-Stokes problem plays an important role in describing the behavior of some non-Newtonian fluids [17]. In this work, we consider the Rayleigh-Stokes problem:

$$\begin{gathered} \partial_t u(x,t) - (1+\kappa\partial_t^{\alpha})\Delta u(x,t) = f(x)\Phi(t), \ (x,t) \in \Omega\times(0,T) \\ u(x,t) = 0, \ \ x \in \partial\Omega, \ \ u(x,0) = u_0(x), \ \ x \in \Omega \\ \int_0^T u(x,t)\mathrm{d}t = g(x), \ \ x \in \Omega \end{gathered} \tag{1}$$

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where

$$\Omega \subset \mathbb{R}^N \ \ (N = 1,2,3), \text{ and } T > 0$$

Here $\kappa > 0$ is a constant, $u_0 \in L^2(\Omega)$, the notations $\partial_t = \partial/\partial t$, and ∂_t^α is the Riemann-Liouville fractional derivative of order $\alpha \in (0, 1)$ defined by [18]:

$$\partial_t^\alpha h(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \xi_{1-\alpha}(t-s)h(s)\mathrm{d}s, \ \ \xi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

In this paper, we used the non-local condition:

$$\int_0^T u(x,t)\mathrm{d}t = g(x), \text{ instead of } u(x,T) = g(x)$$

see in [19]. The couple functions (g, Φ) is approximated by (g, Φ_ϵ) such that

$$\|g - g_\epsilon\|_{L^2(\Omega)} + \|\Phi - \Phi_\epsilon\|_{L^\infty(0,T)} \le \epsilon$$

Until now, very few papers have discussed the problem of investigating the source function for the Rayler-Stokes problem with integral terminal conditions. Therefore, it is a hot topic in the field of inverse problems to deal with ill-posed problems. There are many methods to regularize, typically as: the Tikhonov regularization method, [20, 22], quasi-reversibility method, [23, 24], and quasi boundary value method, [25, 26], the modified quasi boundary value method, [27], the truncation method, [28]. The modified Tikhonov regularization method [29], the fractional Tikhonov regularization method [30, 31], and the simplified Tikhonov regularization method [32]. The contribution of this paper could be summarized as: we verify that *Problem 1* is ill-posed. Next, we construct a regularizing solution using the posterior method of modified iterated Lavrentiev regularization, the ideas of this method for interested readers can be found in digital document [33]. Afterthat, we show the convergent rate between the sought solution and its approximation.

Preliminary

Definition 1. Assume $\{\lambda_p, \phi_p\}$ be the eigenvalues and corresponding eigenvectors of the Laplacian operator $-\Delta$ in Ω. The family of eigenvalues:

$$\{\lambda_p\}_{p=1}^\infty \text{ satisfy } 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_p \le \cdots \text{ where } \lambda_p \to \infty \text{ as } p \to \infty$$

$$\Delta\phi_p(x) = -\lambda_p\phi_p(x), \ \ x \in \Omega$$

$$\phi_p(x) = 0, \ \ x \in \partial\Omega$$

Definition 2. For $\sigma > 0$, defining:

$$\mathbb{H}^\sigma(\Omega) := \left\{ v \in L^2(\Omega), \ \sum_{p=1}^\infty \lambda_p^\sigma \left|\langle v, \phi_p \rangle\right|^2 < +\infty \right\} \tag{2}$$

equipped with the norm

$$\|v\|_{\mathbb{H}^\sigma(\Omega)} = \left(\sum\nolimits_{p=1}^\infty \lambda_p^\sigma \left|\langle v, \phi_p \rangle\right|^2 \right)^{1/2}$$

The solution of problem of eq. (1) is obtained:

$$u(x,t)=\sum_{p=1}^{\infty}\mathcal{K}_p(\alpha,t)\left\langle u_0,\phi_p\right\rangle+\sum_{p=1}^{\infty}\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi(s)\mathrm{d}s\left\langle f,\phi_p\right\rangle\right)\phi_p(x) \tag{3}$$

where

$$F_p(s)=\Phi(s)\left\langle f,\phi_p\right\rangle$$

Here $\mathcal{K}_p(\alpha, t)$ satisfies the equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_p(\alpha,t)+\lambda_p\left(1+\kappa\partial_t^\alpha\right)\mathcal{K}_p(\alpha,t)=0,\ \ t\in(0,T),\ \text{and}\ \ \mathcal{K}_p(\alpha,0)=1 \tag{4}$$

From the condition:

$$\int_0^T u(x,t)\mathrm{d}t=g(x)$$

we get

$$g(x)=\int_0^T u(x,t)\mathrm{d}t=\int_0^T\sum_{p=1}^{\infty}\left\langle f,\phi_p\right\rangle\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi(s)\mathrm{d}s\right)\mathrm{d}t\phi_p(x) \tag{5}$$

A simple calculation gives:

$$f(x)=\sum_{p=1}^{\infty}f_p\phi_p(x)=\sum_{p=1}^{\infty}\frac{\left\langle g,\phi_p\right\rangle}{\displaystyle\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi(s)ds\right)\mathrm{d}t}\phi_p(x) \tag{6}$$

whereby $\mathcal{K}_p$ is defined as in *Lemma 2.1* of [34].

Lemma 1. Assume that:

$$\alpha\in\left(\frac{1}{2},1\right),\ \ \forall t\in[0,T],\ \ \mathcal{K}_p(\alpha,t)\geq\lambda_p^{-1}\tilde{C}(\kappa,\alpha,\lambda_1)$$

This implies that:

$$\int_0^T\mathcal{K}_p(\alpha,T-s)\mathrm{d}s\geq\int_0^T\frac{\tilde{C}(\kappa,\alpha,\lambda_1)}{\lambda_p}\mathrm{d}s=\frac{T\tilde{C}(\kappa,\alpha,\lambda_1)}{\lambda_p} \tag{7}$$

and there exist $\mathcal{M}$ satisfies:

$$\int_0^T\left|\mathcal{K}_p(\alpha,t)\right|^2\mathrm{d}t\leq\frac{\mathcal{M}^2}{\lambda_p^2}\frac{T^{2\alpha-1}}{2\alpha-1} \tag{8}$$

Proof. See in [20].

Lemma 2. Let Φ_0, Φ_1 are positive constants such that

$$\Phi_0\leq\left|\Phi(t)\right|\leq\Phi_1\ \ \forall t\in[0,T]$$

Let choose

$$\epsilon \in \left(0, \frac{\Phi_0}{4}\right), \text{ we obtain } 4^{-1}\Phi_0 \le \left|\Phi_\epsilon\right| \mathcal{C}(\Phi_0, \Phi_1), \text{where } \mathcal{C}(\Phi_0, \Phi_1) = \Phi_1 + \frac{\Phi_0}{4}$$

Proof. The readers can be seen in document [35]. ↪

Lemma 3. [35]. The following inclusions hold true:

$$L^q(\Omega) \to \mathbb{H}^\sigma(\Omega), \text{if } -\frac{N}{4} < \sigma \le 0,\ q \ge \frac{2N}{N-4\sigma},\ \mathbb{H}^\sigma(\Omega) \to L^q(\Omega), \text{if } 0 < \sigma \le \frac{N}{4},\ q \le \frac{2N}{N-4\sigma} \tag{9}$$

The ill-posedness of inverse source problem of eq. (1)

Theorem 1. The inverse source problem of eq. (1) is non-well-posed.

Proof. A linear operator

$$\mathcal{P}: L^2(\Omega) \to L^2(\Omega)$$

is defined:

$$\mathcal{P}f(x) = \sum_{p=1}^{\infty} \left[\int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t \right] \left\langle f, \phi_p \right\rangle \phi_p(x) = \int_\Omega q(x,\omega) f(\omega) \mathrm{d}\omega \tag{10}$$

where

$$q(x,\omega) = \sum_{p=1}^{\infty} \left[\int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t \right] \phi_p(x)\phi_p(\omega)$$

Due to $q(x, \omega) = q(\omega, x)$, we know $\mathcal{P}$ is self-adnoint operator. The $\mathcal{P}_\mathcal{N}$ is the finite rand operators:

$$\mathcal{P}_\mathcal{N} f(x) = \sum_{p=1}^{\mathcal{N}} \left[\int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t \right] \left\langle f, \phi_p \right\rangle \phi_p(x) \tag{11}$$

From eqs. (10) and (11), we have:

$$\left\| \mathcal{P}_\mathcal{N} f - \mathcal{P}f \right\|_{L^2(\Omega)}^2 \le \frac{\mathcal{C}_2^2}{\lambda_\mathcal{N}^2} \sum_{n=\mathcal{N}+1}^{\infty} \left| \left\langle f, \phi_p \right\rangle \right|^2 \text{ where } \mathcal{C}_2^2 = \frac{\mathcal{C}_1^2 \mathcal{M}^2 T^{2\alpha+1}}{2\alpha - 1}$$

Therefore, we have $\| \mathcal{P}_\mathcal{N} - \mathcal{P} \|^2_{L^2(\Omega)} \to 0$ in the sense of operator norm:

$$L\left(L^2(\Omega); L^2(\Omega)\right) \text{ as } \mathcal{N} \to \infty$$

Also, $\mathcal{K}$ is a compact operator. Next, the SVD of $\mathcal{P}$:

$$\Xi_p = \int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t, \text{ with the final data } \ell^k = \frac{\phi_k}{\sqrt{\lambda_k}}$$

By eq. (6), the source term corresponding to ℓ^k:

$$f^k(x) = \sum_{p=1}^{\infty} \left| \int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t \right|^{-1} \left\langle \frac{\phi_k}{\sqrt{\lambda_k}}, \phi_p \right\rangle = \left[\sqrt{\lambda_k} \int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t \right]^{-1} \phi_k$$

With $\ell = 0$ then $f = 0$. An error in L^2-norm between ℓ^k and ℓ:

$$\left\|\ell^k - \ell\right\|_{L^2(\Omega)} = \frac{1}{\sqrt{\lambda_k}} \to \lim_{k\to+\infty}\left\|\ell^k - \ell\right\|_{L^2(\Omega)} = \lim_{k\to+\infty}\left(\frac{1}{\sqrt{\lambda_k}}\right) = 0 \tag{12}$$

The estimate error between f^k and f:

$$\left\|f^k - f\right\|^2_{L^2(\Omega)} = \lambda_k^{-1}\left|\int_0^T\left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s\right)\mathrm{d}t\right|^{-2}$$

Due to:

$$\left|\int_0^T\left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s\right)\mathrm{d}t\right|^{2} \le \frac{\mathcal{C}_2^2}{\lambda_k^2}$$

For any $\alpha > 1/2$, we obtain:

$$\left\|f^k - f\right\|^2_{L^2(\Omega)} \ge \frac{\lambda_k}{\mathcal{C}_2^2} \text{ this leads to } \lim_{k\to+\infty}\frac{\sqrt{\lambda_k}}{\mathcal{C}_2} = + \tag{13}$$

Combining eqs. (12) and (13), the inverse source problem of eq. (1) is ill-posed.

Conditional stability of source term *f*

Theorem 2. Let:

$$f \in \mathbb{H}^\sigma(\Omega) \text{ such that } \|f\|_{\mathbb{H}^\sigma(\Omega)} \le E$$

for $E > 0$, then it gives

$$\|f\|_{L^2(\Omega)} \le \mathcal{D}(\sigma, T)E^{\frac{1}{\sigma+1}}\|g\|^{\frac{\sigma}{\sigma+1}}_{L^2(\Omega)}, \text{ where } \mathcal{D}(\sigma,T) = \left(\mathcal{C}_0^{\frac{\sigma}{\sigma+1}} T^{\frac{\sigma}{\sigma+1}} \tilde{C}^{\frac{\sigma}{\sigma+1}}(\kappa, \alpha, \lambda_1)\right)^{-1} \tag{14}$$

Proof. See in [20].

The modified Larentiev regularization method for problem of eq. (1)

In this subsection, the modified Larentiev regularization method is considered and gives the convergent rate, we denote the noise measurement of (Φ, g) as $(\Phi_\epsilon, g_\epsilon)$. Based on the [33], form now on, for a shorter, we denote:

$$\mathcal{Q}_p(\alpha, \Phi_\epsilon) = \int_0^T\left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi_\epsilon(s)\mathrm{d}s\right)\mathrm{d}t$$

the modified iterated Lavrentiev is introduced:

$$f_{\epsilon,0}(x) = 0,\ \left(\mathcal{Q}_p(\alpha, \Phi_\epsilon) + \beta\right) f_{\epsilon,a}(x) = \beta\, f_{\epsilon,a-1}(x) + g_\epsilon(x) \tag{15}$$

where a is an iterative step number. From eq. (15), we find that:

$$f_{\epsilon,a}(x) = \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)}\right) f_{\epsilon,a-1}(x) + \frac{g_\epsilon(x)}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \tag{16}$$

This implies that:

$$f_{\epsilon,a}(x)=\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^a f_{\epsilon,0}(x)+\frac{g(x)}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\sum_{k=1}^{a-1}\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^k=$$
$$=\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^a f_{\epsilon,0}(x)+\left(1-\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^a\right)\frac{g_{\grave{o}}(x)}{\mathcal{Q}_p(\alpha,\Phi_\epsilon)} \tag{17}$$

By condition (15), we know that:

$$f_{\epsilon,a}(x)=\left[1-\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^a\right]\frac{g_\epsilon(x)}{\mathcal{Q}_p(\alpha,\Phi_\epsilon)} \tag{18}$$

Now, here we denote $a^\dagger$ satisfies:

$$1<a^\dagger=a^\dagger(\epsilon)$$

is the first iterative step which satisfies the inequality eq. (19) and ζ_ϵ is defined later. We consider the discrepancy principle:

$$\left\|\mathcal{Q}_p(\alpha,\Phi_\epsilon)\,f_{\epsilon,a^\dagger}-g_\epsilon\right\|_{L^2(\Omega)}\le\zeta\epsilon<\left\|\mathcal{Q}_p(\alpha,\Phi_\epsilon)\,f_{\epsilon,a^\dagger-1}-g_\epsilon\right\|_{L^2(\Omega)} \tag{19}$$

Next, we have the convergent rate for the modified iterated Lavrentiev eq. (18) solution and the sought solution (6).

Lemma 4. Assume that:

$$g_\epsilon, g\in L^2(\Omega)\ \text{ such that }\ \|g_\epsilon-g\|_{L^2(\Omega)}\le\epsilon\ \text{ and }\ f\in\mathbb{H}^\sigma(\Omega)$$

then:

$$a^\dagger\le(\mathcal{R}E)^{\frac{2}{\sigma+2}}\left[(\zeta-1)\epsilon\right]^{-\frac{2}{\sigma+2}} \tag{20}$$

Proof. By eq. (18), we have:

$$\left\|\mathcal{Q}_p(\alpha,\Phi_\epsilon)\,f_{\epsilon,a^\dagger-1}-g_\epsilon\right\|_{L^2(\Omega)}=\left\|\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^{a^\dagger-1}g_\epsilon\right\|_{L^2(\Omega)} \tag{21}$$

The right hand side of eq. (21) can be bounded:

$$\left\|\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^{a^\dagger-1}g_\epsilon\right\|_{L^2(\Omega)}=\left\|\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^{a^\dagger-1}(g_\epsilon-g)\right\|_{L^2(\Omega)} \tag{22}$$

Let us evaluate $\mathcal{K}_{1,2}$: step by step.

Estimate of $\mathcal{K}_1$:

$$\mathcal{K}_1:=\left\|\left(\frac{\beta}{\beta+\mathcal{Q}_p(\alpha,\Phi_\epsilon)}\right)^{a^\dagger-1}(g_\epsilon-g)\right\|_{L^2(\Omega)}\le\epsilon$$

Estimate of $\mathcal{K}_2$:

$$\mathcal{K}_2 := \left\| \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger - 1} \lambda_p^{-\frac{\sigma}{2}} \mathcal{Q}_p(\alpha, \Phi) \frac{\lambda_p^{\frac{\sigma}{2}} g(\cdot)}{\mathcal{Q}_p(\alpha, \Phi)} \right\|_{L^2(\Omega)} \le \mathcal{K}_* E$$

$$\mathcal{K}_* = \sup_{p \ge 1} \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger - 1} \lambda_p^{-\frac{\sigma}{2}} \mathcal{Q}_p(\alpha, \Phi) \le \sup_{p \ge 1} \left(\frac{\beta}{\beta + \lambda_p^{-1} 4^{-1} \mathcal{C}_0 T \tilde{C}(\kappa, \alpha, \lambda_1)} \right)^{a^\dagger - 1} \lambda_p^{-\frac{\sigma}{2} - 1} \mathcal{C}_1 \mathcal{C}_2 \quad (23)$$

Putting:

$$\nu = 4^{-1} \mathcal{C}_0 T \tilde{C}(\kappa, \alpha, \lambda_1)$$

let us define the function $\mathcal{S}(\lambda_p)$

$$\mathcal{S}(\lambda_p) = \left(\frac{\beta}{\beta + \frac{\nu}{\lambda_p}} \right)^{a^\dagger - 1} \lambda_p^{-\frac{\sigma}{2} - 1} = \left(1 - \frac{\nu}{\beta \lambda_p + \nu} \right)^{a^\dagger - 1} \lambda_p^{-\frac{\sigma}{2} - 1} \le \left(1 - \frac{\mathcal{V}}{2 \max\{\beta, \mathcal{V}\} \lambda_p} \right)^{a^\dagger - 1} \lambda_p^{-\frac{\sigma}{2} - 1} \quad (24)$$

where $\lambda_p = x$, we have the function

$$\mathcal{Z}(x) = \left(1 - \frac{\nu}{2 \max\{\beta, \nu\} x} \right)^{a^\dagger - 1} x^{-\frac{\sigma}{2} - 1}$$

Taking the derivative of $\mathcal{Z}(x)$, then we find:

$$x_0 = \frac{2(a^\dagger - 1)\nu}{(\sigma + 2) 2 \max\{\beta, \nu\}}$$

As the point we see, this leads:

$$\mathcal{Z}(x_0) \le \left[\frac{2(a^\dagger - 1)\nu}{2 \max\{\beta, \nu\} (\sigma + 2)} \right]^{-\frac{\sigma}{2} - 1}$$

using the fact that

$$2(a^\dagger - 1) \ge a^\dagger$$

we can find that

$$\mathcal{Z}(x_0) \le \left(\frac{\nu}{2 \max\{\beta, \nu\} (\sigma + 2)} \right)^{-\frac{\sigma}{2} - 1} (a^\dagger)^{-\frac{\sigma}{2} - 2}$$

this implies that

$$\mathcal{K}_2 \le \mathcal{R} E (a^\dagger)^{-\frac{\sigma + 2}{2}}$$

where by $\mathcal{R}$ is defined:

$$\mathcal{R} = \left(\frac{\nu}{2 \max\{\beta, \nu\} (\sigma + 2)} \right)^{-\frac{\sigma}{2} - 1} \mathcal{C}_1 \mathcal{C}_2$$

We obtain

$$\left\| \mathcal{Q}_p(\alpha,\Phi_\epsilon) f_{\epsilon,a^\dagger-1} - g_\epsilon \right\|_{L^2(\Omega)} \leq \epsilon + \mathcal{R}E\left(a^\dagger\right)^{-\frac{\sigma+2}{2}} \tag{25}$$

Thank to eq. (19), we have:

$$\zeta\epsilon \leq \epsilon + \mathcal{R}E(a^\dagger)^{-\frac{\sigma+2}{2}} \text{ this implies that } (\zeta-1)\epsilon \leq \mathcal{R}E(a^\dagger)^{-\frac{\sigma+2}{2}}$$

It gives:

$$(a^\dagger)^{\frac{\sigma+2}{2}} \leq \frac{\mathcal{R}E}{(\zeta-1)\epsilon} \text{ this leads to } a^\dagger \leq \left(\frac{\mathcal{R}E}{(\zeta-1)\epsilon}\right)^{\frac{2}{\sigma+2}} \tag{26}$$

The proof of this *Lemma* is completed.

Theorem 3. For $\sigma > 0$, assume that $f \in \mathbb{H}^\sigma(\Omega)$ such that $\|f\|_{\mathbb{H}^\sigma(\Omega)} \leq E$. Let:

$$g, g_\epsilon \in L^2(\Omega) \text{ such that } \| g_\epsilon - g \|_{L^2(\Omega)} \leq \epsilon$$

and using the estimate $a^\dagger$ in the *Lemma 4*, we get:

$$\left\| f_{\epsilon,a^\dagger} - f \right\|_{L^2(\Omega)} \leq \max\left\{ \epsilon, \epsilon^{\frac{\sigma}{\sigma+2}}, \epsilon^{\frac{\sigma}{(\sigma+1)(\sigma+2)}} \right\} \tag{27}$$

As a consequence, for ϵ tends to 0, we have:

$$\left\| f_{\epsilon,a^\dagger} - f \right\|_{L^2(\Omega)} \to 0$$

Proof. Defined the function $f_{a^\dagger}(x)$ is given by:

$$f_{a^\dagger}(x) = \sum_{p=1}^{\infty} \left[1 - \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi)} \right)^{a^\dagger} \right] \frac{\langle g, \phi_p \rangle}{\mathcal{Q}_p(\alpha,\Phi)} \tag{28}$$

By the triangle inequality, we can find that:

$$\left\| f_{\epsilon,a^\dagger} - f \right\|_{L^2(\Omega)} \leq \underbrace{\left\| f_{\epsilon,a^\dagger} - f_{a^\dagger} \right\|_{L^2(\Omega)}}_{\text{first term}} + \underbrace{\left\| f_{a^\dagger} - f \right\|_{L^2(\Omega)}}_{\text{second term}} \tag{29}$$

We have an estimate for the second term in eq. (29), we have:

$$\left\| \Xi_p \left(f_{a^\dagger} - f \right) \right\|_{L^2(\Omega)} = \left\| \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi)} \right)^{a^\dagger} \langle g, \phi_p \rangle \right\|_{L^2(\Omega)} \leq \chi_1 + \chi_2 + \chi_3 \tag{30}$$

where by:

$$\chi_1 = \left\| \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi)} \right)^{a^\dagger} - \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi_\epsilon)} \right)^{a^\dagger} \langle g, \phi_p \rangle \right\|_{L^2(\Omega)}$$

$$\chi_2 = \left\| \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi_\epsilon)} \right)^{a^\dagger} \left| \langle g - g_\epsilon, \phi_p \rangle \right| \right\|_{L^2(\Omega)}, \quad \chi_3 = \left\| \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi_\epsilon)} \right)^{a^\dagger} \left| \langle g_\epsilon, \phi_p \rangle \right| \right\|_{L^2(\Omega)} \tag{31}$$

Let us evaluate X_i, $i = 1, 2, 3$, step by step.

Estimate of X_i: By the inequality:

$$|y_1^c - y_2^c| \leq c\,|y_1 - y_2|,\quad c \geq 1,\ \ y_1, y_2 \in [0,1]$$

and the boundary condition $f \in L^2(\Omega)$, we obtain that:

$$\begin{aligned} X_1 &\leq \| a^\dagger \,|\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi)} - \frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi_\epsilon)}| < g, \phi_p > \|_{L^2(\Omega)} \\ &\leq a^\dagger \sup_{p\geq 1} \left| \frac{\beta\, \mathcal{Q}_p(\alpha, \Phi - \Phi_\epsilon)}{(\beta + \mathcal{Q}_p(\alpha,\Phi)) \times (\beta + \mathcal{Q}_p(\alpha,\Phi_\epsilon))} \right| \| g \|_{L^2(\Omega)} \\ &\leq \beta^{-1} \epsilon a^\dagger \| g \|_{L^2(\Omega)} \sup_{p\geq 1} |\mathcal{Q}_p(\alpha)| \\ &\leq \beta^{-1} a^\dagger \epsilon \| g \|_{L^2(\Omega)} \mathcal{C}_2 \lambda_1^{-1} \end{aligned} \tag{32}$$

Thank to eq. (26), we can find that:

$$X_1 \leq \beta^{-1}\epsilon \| g \|_{L^2(\Omega)} \left[\frac{\mathcal{R}E}{(\zeta - 1)\epsilon} \right]^{\frac{2}{\sigma+2}} \mathcal{C}_2 \lambda_1^{-1} \tag{33}$$

Estimate of X_2 and Estimate of X_3 assessed is as simple as follow, by

$$\left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi)} \right)^{a^\dagger} \leq 1$$

and in the view of the stopping rule eq. (19), we conclude:

$$X_2 \leq \epsilon \quad \text{and} \quad X_3 \leq \zeta\epsilon \tag{34}$$

Combining eqs. (29), (33), and (34), we receive:

$$\left\| f_{a^\dagger} - f \right\|_{L^2(\Omega)} \leq \epsilon \left(\|g\|_{L^2(\Omega)} (\mathcal{R}E)^{\frac{2}{\sigma+2}} \left[(\zeta - 1)\epsilon \right]^{-\frac{2}{\sigma+2}} \mathcal{C}_2 \lambda_1^{-1} + (\zeta + 1) \right) \tag{35}$$

Thank to a priori bound condition $f \in \mathbb{H}^\sigma(\Omega)$ one has:

$$\left\| \Xi_p (f_{\epsilon,a^\dagger} - f) \right\|_{L^2(\Omega)} \leq \left\| \sum_{p=1}^{\infty} \lambda_p^{\frac{\sigma}{2}} \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha,\Phi)} \right)^{a^\dagger} \frac{\langle g, \phi_p \rangle}{\mathcal{Q}_p(\alpha,\Phi)} \right\|_{L^2(\Omega)} \leq E \tag{36}$$

We deduce:

$$\left\| f_{a^\dagger} - f \right\|_{L^2(\Omega)} \leq \epsilon^{\frac{\sigma}{\sigma+1}} E^{\frac{1}{\sigma+1}} \left(\|g\|_{L^2(\Omega)} \left[\frac{\mathcal{R}E}{(\zeta - 1)\epsilon} \right]^{\frac{2}{\sigma+2}} \mathcal{C}_2 \lambda_1^{-1} + (\zeta + 1) \right)^{\frac{\sigma}{\sigma+1}} \mathcal{D}(\sigma, T) \tag{37}$$

On the other hand, estimating the first term of eq. (29), we get:

$$\left\| f_{\epsilon,a^\dagger} - f_{a^\dagger} \right\|_{L^2(\Omega)} \leq \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 \tag{38}$$

where

$$\mathcal{S}_1 = \left\| \left[1 - \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger} \right] \frac{\left| \left\langle g_\epsilon - g, \phi_p \right\rangle \right|}{\mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right\|_{L^2(\Omega)}$$

$$\mathcal{S}_2 = \left\| \left[1 - \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger} \right] \left(\frac{\left| \left\langle g, \phi_p \right\rangle \right|}{\mathcal{Q}_p(\alpha, \Phi)} - \frac{\left| \left\langle g, \phi_p \right\rangle \right|}{\mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right) \right\|_{L^2(\Omega)} \tag{39}$$

$$\mathcal{S}_3 = \left\| \left[\left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi)} \right)^{a^\dagger} - \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger} \right] \frac{\left| \left\langle g, \phi_p \right\rangle \right|}{\mathcal{Q}_p(\alpha, \Phi)} \right\|_{L^2(\Omega)}$$

Let us evaluate $\mathcal{S}_j$, $j = 1, 2,$, we obtain that:

Estimate of $\mathcal{S}_1$, we have:

$$\mathcal{S}_1 \le \sup_{p \ge 1} \frac{1 - \left(1 - \frac{\mathcal{Q}_p(\alpha, \Phi_\epsilon)}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger}}{\mathcal{Q}_p(\alpha, \Phi_\epsilon)} \left| \left\langle g_\epsilon - g, \phi_p \right\rangle \right| \le \epsilon \sup_{p \ge 1} \frac{1 - \left(1 - \frac{\mathcal{Q}_p(\alpha, \Phi_\epsilon)}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger}}{\mathcal{Q}_p(\alpha, \Phi_\epsilon)} \tag{40}$$

Due to Bernoulli inequality, we have

$$\left(1 - \frac{\mathcal{Q}_p(\alpha, \Phi_\epsilon)}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger} \ge 1 - a^\dagger \left(\frac{\mathcal{Q}_p(\alpha, \Phi_\epsilon)}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)$$

This is to show us that:

$$\left[\mathcal{Q}_p(\alpha, \Phi_\epsilon) \right]^{-1} \left[1 - \left(1 - \frac{\mathcal{Q}_p(\alpha, \Phi_\epsilon)}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger} \right] \le \frac{a^\dagger}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \le \frac{a^\dagger}{\beta}, \text{ this leads to } \mathcal{S}_1 \le \frac{\epsilon a^\dagger}{\beta} \tag{41}$$

Estimate of $\mathcal{S}_1$, $\mathcal{S}_2$ can be bounded:

$$\mathcal{S}_2 \le \left\| \left[1 - \left(\frac{\beta}{\beta + \mathcal{Q}_p(\alpha, \Phi_\epsilon)} \right)^{a^\dagger} \right] \left(\frac{\mathcal{Q}_p(\alpha, \Phi_\epsilon - \Phi)}{\mathcal{Q}_p(\alpha, \Phi_\epsilon)} \frac{\left| \left\langle g, \phi_p \right\rangle \right|}{\mathcal{Q}_p(\alpha, \Phi)} \right) \right\|_{L^2(\Omega)} \le \frac{\epsilon a^\dagger}{\beta} \frac{\| f \|_{L^2(\Omega)}}{\mathcal{C}_0} \tag{42}$$

Estimate of $\mathcal{S}_3$, using the same proof eq. (33) and condition $f \in \mathbb{L}^2(\Omega)$, we have:

$$\mathcal{S}_3 \le \frac{\epsilon \| f \|_{L^2(\Omega)}}{\beta} \left(\frac{\mathcal{R}E}{(\zeta - 1)\epsilon} \right)^{\frac{2}{\sigma + 2}} \mathcal{C}_2 \lambda_1^{-1} \tag{43}$$

Combining eqs. (41)-(43), we have:

$$\left\| f_{\epsilon, a^\dagger} - f_{a^\dagger} \right\|_{L^2(\Omega)} \le 2 \frac{\epsilon a^\dagger}{\beta} \max \left\{ 1, \frac{\|g\|_{L^2(\Omega)}}{\mathcal{C}_0} \right\} + \frac{\epsilon \| f \|_{L^2(\Omega)}}{\beta} \left(\frac{\mathcal{R}E}{(\zeta - 1)\dot{o}} \right)^{\frac{2}{\sigma + 2}} \mathcal{C}_2 \lambda_1^{-1} \tag{44}$$

Therefore, by eqs. (37) and (44), we conclude:

$$\left\| f_{\epsilon,a^{\dagger}} - f \right\|_{L^2(\Omega)} \le 2\beta^{-1}\epsilon a^{\dagger} \max\left\{1, \frac{\| g \|_{L^2(\Omega)}}{\mathcal{C}_0}\right\} + \epsilon^{\frac{\sigma}{\sigma+2}} \|f\|_{L^2(\Omega)} \beta^{-1} \left(\frac{\mathcal{R}E}{\zeta - 1}\right)^{\frac{2}{\sigma+2}} \mathcal{C}_2 \lambda_1^{-1} \tag{45}$$

The provision of this *Theorem* is completed.

Regularization with the estimate in L^q paces

In this subsection, let g_ϵ is observation data and satisfied that:

$$\| g_\epsilon - g \|_{L^q(\Omega)} \le \epsilon \tag{46}$$

Theorem 4. Let g_ϵ be as in eq. (46) and f belongs to $\mathbb{H}^\sigma(\Omega)$ for any $\sigma > 0$. We get a regularized solution:

$$f_\epsilon^{\mathcal{P}_\epsilon}(x) = \sum_{p=1}^{\mathcal{P}_\epsilon} \frac{\langle g, \phi_p \rangle \phi_p(x)}{\int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t}, \text{ and } f^{\mathcal{P}_\epsilon}(x) = \sum_{p=1}^{\mathcal{P}_\epsilon} \frac{\langle g, \phi_p \rangle \phi_p(x)}{\int_0^T \left(\int_0^t \mathcal{K}_p(\alpha, t-s)\Phi(s)\mathrm{d}s \right) \mathrm{d}t} \tag{47}$$

By choosing:

$$\mathcal{P}_\epsilon = \left(\frac{1}{\epsilon}\right)^{\frac{1-s}{q-\sigma+1}}, \ 0 < s < 1$$

in which:

$$-\frac{N}{4} < q \le \min\left\{0, \frac{(q-2)N}{4q}\right\}, \ 0 \le \sigma < \frac{N}{4} \tag{48}$$

then we get

$$\| f_\epsilon^{\mathcal{P}_\epsilon} - f \|_{L^{\frac{2N}{N-4\sigma}}(\Omega)} \to 0 \text{ is of order} \max\left\{\epsilon^s, \epsilon, \epsilon^{\frac{\sigma(1-s)}{q-\sigma+1}}\right\} \tag{49}$$

Proof. Since the Sobolev embedding

$$L^q(\Omega) \rightarrowtail \mathbb{H}^\sigma(\Omega)$$

then a exists constant $\mathcal{C}(q,\sigma)$ such that:

$$\| g_\epsilon - g \|_{\mathbb{H}^\sigma(\Omega)} \le \mathcal{C}(q,\sigma) \| g_\epsilon - g \|_{L^q(\Omega)} \le \mathcal{C}(q,\sigma)\epsilon \tag{50}$$

For $\sigma > 0$, we have:

$$\| f_\epsilon^{\mathcal{P}_\epsilon} - f \|_{\mathbb{H}^\sigma(\Omega)} \le \| f_\epsilon^{\mathcal{P}_\epsilon} - f^{\mathcal{P}_\epsilon} \|_{\mathbb{H}^\sigma(\Omega)} + \| f^{\mathcal{P}_\epsilon} - f \|_{\mathbb{H}^\sigma(\Omega)} \tag{51}$$

We consider the term:

$$\| f_\epsilon^{\mathcal{P}_\epsilon} - f \|_{\mathbb{H}^\sigma(\Omega)} \text{ for any } 0 < \sigma < \frac{N}{4}$$

Indeed, we get:

$$f_\epsilon^{\mathcal{P}_\epsilon}(x)-f^{\mathcal{P}_\epsilon}(x)=\sum_{p=1}^{\mathcal{P}_\epsilon}\frac{\left\langle g_\epsilon-g,\phi_p\right\rangle\phi_p(x)}{\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi_\epsilon(s)\mathrm{d}s\right)\mathrm{d}t}+$$
$$+\sum_{p=1}^{\mathcal{P}_\epsilon}\left[\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi_\epsilon(s)\mathrm{d}s\right)\mathrm{d}t\right|^{-1}-\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi(s)\mathrm{d}s\right)\mathrm{d}t\right|^{-1}\right]\left\langle g,\phi_p\right\rangle\phi_p(x) \tag{52}$$

Next, the estimate error of

$$\left\|f-f^{\mathcal{P}_\epsilon}\right\|_{\mathbb{H}^\sigma(\Omega)}$$

we receive:

$$\left\|f-f^{\mathcal{P}_\epsilon}\right\|_{\mathbb{H}^\sigma(\Omega)}^2\le\sum_{p=\mathcal{P}_\epsilon+1}^{\infty}\lambda_p^{-2\sigma}\lambda_p^{2\sigma}\frac{\left|\left\langle g,\phi_p\right\rangle\right|^2}{\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi(s)\mathrm{d}s\right)\mathrm{d}t\right|^2}\le\sum_{p=\mathcal{P}_\epsilon+1}^{\infty}\lambda_p^{-2\sigma}\lambda_p^{2\sigma}\left|\left\langle f,e_p\right\rangle\right|^2\le$$
$$\le(\mathcal{P}_\epsilon)^{-2\sigma}\sum_{p=\mathcal{P}_\epsilon+1}^{\infty}\lambda_p^{2\sigma}\left|f,\phi_p\right|^2\le(\mathcal{P}_\epsilon)^{-2\sigma}\left\|f\right\|_{\mathbb{H}^\sigma(\Omega)}^2 \tag{53}$$

From eq. (51), we can know:

$$\left\|f_\epsilon^{\mathcal{P}_\epsilon}-f^{\mathcal{P}_\epsilon}\right\|_{\mathbb{H}^\sigma(\Omega)}^2\le2\sum_{p=1}^{\mathcal{P}_\epsilon}\lambda_p^{2q-2\sigma}\frac{\lambda_p^{2\sigma}\left|\left\langle g_\epsilon-g,\phi_p\right\rangle\right|^2}{\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi_\epsilon(s)\mathrm{d}s\right)\mathrm{d}t\right|^2}+$$
$$+2\sum_{p=1}^{\mathcal{P}_\epsilon}\frac{\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)(\Phi_\epsilon(s)-\Phi(s)\mathrm{d}s\right)\mathrm{d}t\right|^2}{\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi_\epsilon(s)\mathrm{d}s\right)\mathrm{d}t\right|^2}\frac{\lambda_p^{2\sigma}\left|\left\langle g,\phi_p\right\rangle\right|^2}{\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi(s)\mathrm{d}s\right)\mathrm{d}t\right|^2}\le$$
$$\le2\sum_{p=1}^{\mathcal{P}_\epsilon}\lambda_p^{2q-2\sigma+2}\frac{\lambda_p^{2q}\left|\left\langle g_{\dot{o}}-g,\phi_p\right\rangle\right|^2}{|T\tilde{C}(\kappa,\alpha,\lambda_1)|^2}+\frac{32\left\|\Phi_\epsilon-\Phi\right\|_{L^\infty(0,T)}^2}{\Phi_1^2}\sum_{p=1}^{\mathcal{P}}\frac{\lambda_p^{2\sigma}\,|\left|\left\langle g,\phi_p\right\rangle\right|^2}{\left|\int_0^T\left(\int_0^t\mathcal{K}_p(\alpha,t-s)\Phi(s)ds\right)dt\right|^2}\le \tag{54}$$
$$\le2\sum_{p=1}^{\mathcal{P}_\epsilon}\lambda_p^{2q-2\sigma+2}\frac{\lambda_p^{2\sigma}\left\|g_\epsilon-g\right\|_{\mathbb{H}^q(\Omega)}^2}{\left|T\tilde{C}(\kappa,\alpha,\lambda_1)\right|^2}+\frac{32\left\|\Phi_\epsilon-\Phi\right\|_{L^\infty(0,T)}^2}{\Phi_1^2}\sum_{p=1}^{\infty}\lambda_p^{2\sigma}\left|\left\langle f,\phi_p\right\rangle\right|^2$$

Using the condition in eq. (50), we can know:

$$\| f_{\epsilon}^{\mathcal{P}_\epsilon} - f^{\mathcal{P}_\epsilon} \|^2_{\mathbb{H}^\sigma(\Omega)} \le \frac{2\mathcal{C}^2(q,\sigma)\epsilon^2}{|T\tilde{C}(\kappa,\alpha,\lambda_1)|^2}(\mathcal{P}_\epsilon)^{2q-2\sigma+2} + 32\left(\frac{\epsilon}{\Phi_1}\right)^2 \| f \|^2_{\mathbb{H}^\sigma(\Omega)} \tag{55}$$

Due to Sobolev embedding:

$$\mathbb{H}^\sigma(\Omega) \to L^{\frac{2N}{N-4\sigma}}(\Omega)$$

combining eqs. (51)-(53), we conclude that:

$$\left\| f_\epsilon^{\mathcal{P}_\epsilon} - f \right\|_{L^{\frac{2N}{N-4\sigma}}(\Omega)} \le \mathcal{C}(N,\sigma)\left\| f_\epsilon^{\mathcal{P}_\epsilon} - f \right\|_{\mathbb{H}^\sigma(\Omega)} \le \mathcal{C}(N,\sigma)\left\| f_\epsilon^{\mathcal{P}_\epsilon} - f^{\mathcal{P}_\epsilon} \right\|_{\mathbb{H}^\sigma(\Omega)} +$$

$$+\mathcal{C}(N,\sigma) \| f^{\mathcal{P}_\epsilon} - f \|_{\mathbb{H}^\sigma(\Omega)} \le \frac{\sqrt{2}\mathcal{C}_{q,\sigma}\epsilon}{|T\tilde{C}(\kappa,\alpha,\lambda_1)|}(\mathcal{P}_\epsilon)^{q-\sigma+1} + 4\sqrt{2}\mathcal{C}(N,\sigma)\left(\frac{\epsilon}{\Phi_1}\right)\|f\|_{\mathbb{H}^\sigma(\Omega)} +$$

$$+\mathcal{C}(N,\sigma)(\mathcal{P}_\epsilon)^{-\sigma}\|f\|_{\mathbb{H}^\sigma(\Omega)} \le \epsilon^s \frac{\sqrt{2}\mathcal{C}_{q,\sigma}}{|T\tilde{C}(\kappa,\alpha,\lambda_1)|} + \mathcal{C}(N,\sigma)\left(\frac{4\sqrt{2}}{\Phi_1}\epsilon + \epsilon^{\frac{\sigma(1-s)}{q-\sigma+1}}\right)\|f\|_{\mathbb{H}^\sigma(\Omega)} \tag{56}$$

Conclusion

This study examines the Rayleigh-Stokes equation's inverse source issue. The regularized solution was developed using the modified Lavrentive regularization approach. After that, we considered the convergent rate between the exact solution and its approximation.

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