VARIABLE COEFFICIENT KdV SYSTEM FOR THE NON-LINEAR ROSSBY WAVES

by

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The paper deals with a non-linear barotropic vorticity equation in a shear flow, the model is applied to get the non-linear Rossby solitary wave with the effects of topography and other external forcing. Basing on the perturbation method, the variable coefficient KdV equation is derived for Rossby waves. The periodic-like solution for the equation is obtained with the help of Jacobi elliptic functions, the solitary solutions can also be obtained in the limit case. The analysis indicates that the wave amplitude and velocity will be related to the topography effect. It is also shown that the external forcing plays an important role in evolution of the waves.

Key words: *Rossby solitary wave, Jacobi elliptic function, variable coefficient KdV*

Introduction

The theoretical analysis of non-linear Rossby waves has attracted much attention in the rotational fluid in the evolution of the large scale atmospheric and ocean dynamic. The variable coefficient non-linear models would be suit for explaining some non-linear problems in the atmosphere and the ocean, the extended variable coefficient (VC), KdV (VC-KdV) equation under an external forcing was derived for large amplitude equatorial Rossby solitary wave in a shear flow [1], Tang *et al.* [2] derived the variable coefficient non-linear systems including VC-KdV, VC-mKdV, and VC-NLS equations.

It is generally believed that the models include topography, friction, dissipation, and exogenous factors in real ocean and atmosphere [3-5], so many models are difficult to solve due to the non-linearity. Therefore, it is desired to get the exact solutions of non-linear equations. Up to now, people have obtained several sorts of solutions for various evolution equations, including the solitary and periodic solutions [6-8], M-lump solution, and N-soliton [9, 10]. In fact, some solutions to the VC equations are obtained with the help of Jacobi elliptic functions for the non-linear Rossby solitary waves [11, 12]. In this paper, we prepare to derive VC non-linear systems basing on quasi-geostrophic barotropic model, and find the theoretical solutions which might be applied to explain some phenomena that occur in ocean and atmosphere.

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The derivation of the governing equations

Based on the quasi-geostrophic barotropic model in the paper of Pedlosky [13], it is given the vorticity equation including the topography, and external source is:

$$\left(\frac{\partial}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\right)[f+\nabla^2\Psi+\frac{f_0}{H}h_b(y)]=Q(y,t) \tag{1}$$

where $f = f_0 + \beta_0 y$ is the Coriolis parameter, β_0 – a constant, Ψ – the stream function, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ – the Laplace operator, $h_b(y)$ – the topography which is taken to be a function of *y*, and *Q* – the external source.

We make non-dimensionalization on eq. (1)

$$(x,y)=L(x^*,y^*)\,,\ t=\frac{L}{U}t^*,\ (u,v)=U(u^*,v^*)\,,\ \Psi=UL\Psi^*$$

$$Q=\mu\frac{U}{L}Q^*,\ f=f_0+\beta_0 y=f_0+\frac{U}{L^2}\beta^* L y^*,\ h_b=\frac{H}{f_0}\frac{U}{L}h_b^* \tag{2}$$

where μ is a small number.

Substituting eq. (2) into eq. (1), and omit the asterisk, then the non-dimensional equation is:

$$\left(\frac{\partial}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\right)[\beta y+\nabla^2\Psi+h_b(y)]=\mu Q(y,t) \tag{3}$$

The side boundary condition of non-dimensional form is:

$$\frac{\partial\Psi}{\partial x}=0,\ y=0,1 \tag{4}$$

We assume that the stream function form is:

$$\Psi(x,y,t)=-\int_0^y[u(y)-c_0]\mathrm{d}y+\varepsilon\psi(x,y,t) \tag{5}$$

where $u(y)$ is the shearing zonal flow, c_0 is equals to the phase speed of linear long wave in the shearing flow, $\psi(x,y,t)$ – the disturbed stream function, and $\varepsilon \ll 1$ – a small magnitude parameter. Substituting eq. (5) into eq. (3), we derive the equation regarding the disturbed stream function:

$$\varepsilon\frac{\partial\nabla^2\psi}{\partial t}+\varepsilon[u(y)-c_0]\frac{\partial\nabla^2\psi}{\partial x}+\varepsilon\frac{\partial\psi}{\partial x}[h_b'(y)+\beta-u''(y)]+\varepsilon^2 J[\psi,\nabla^2\psi]=\mu Q(y,t) \tag{6}$$

where $J[a,b]=\dfrac{\partial a}{\partial x}\dfrac{\partial b}{\partial y}-\dfrac{\partial a}{\partial y}\dfrac{\partial b}{\partial x}$.

We introduce the following stretching transformation:

$$X=\varepsilon^{\frac{1}{2}}x,\quad T=\varepsilon^{\frac{3}{2}}t \tag{7}$$

If the external forcing is weak:

$$\mu Q(y,t)=\varepsilon^{5/2}Q_1(y,T) \tag{8}$$

substituting eqs. (7) and (8) into eq. (6):

$$\frac{\partial\psi}{\partial X}[h_0'(y)+\beta-u''(y)]+[u(y)-c_0]\frac{\partial}{\partial X}\frac{\partial^2\psi}{\partial y^2}+\varepsilon[(u(y)-c_0]\frac{\partial^3\psi}{\partial X^3}+$$
$$+\frac{\partial}{\partial T}\frac{\partial^2\psi}{\partial y^2}+J\left(\psi,\frac{\partial^2\psi}{\partial y^2}\right)=\varepsilon Q_1(y,T) \tag{9}$$

where $J[a,b]=\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$.

The disturbed stream function has small parameter expansion formula:

$$\psi=\psi_0(X,y,T)+\varepsilon\psi_1(X,y,T)+\varepsilon^2\psi_2(X,y,T)+\cdots \tag{10}$$

substituting eq. (10) into eq. (9) leads to:

ε^0 order

$$[u(y)-c_0]\frac{\partial^2}{\partial y^2}\frac{\partial\psi_0}{\partial X}+[h_0'(y)+\beta-u''(y)]\frac{\partial\psi_0}{\partial X}=0 \tag{11}$$

$$\frac{\partial\psi_0}{\partial X}=0,\quad y=0,1 \tag{12}$$

Equation (11) has the following variable-separation solution:

$$\psi_0=A(X,T)\phi_0(y,T) \tag{13}$$

applying eq. (13) to eqs. (11) and (12) leads to:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{h_0'(y)+\beta-u''(y)}{u(y)-c_0}\right]\phi_0(y,T)=0 \tag{14}$$

$$\phi_0\big|_{y=0,1}\to 0 \tag{15}$$

in order to discuss the amplitude $A(X,T)$, it is need high order equation to solve.

ε^1 order

$$L_0[\psi_1]=-\left\{L_1[\psi_0]+J\left[\psi_0,\frac{\partial^2\psi_0}{\partial y^2}\right]\right\}+Q_1 \tag{16}$$

$$\frac{\partial\psi_1}{\partial X}=0,\quad y=0,1 \tag{17}$$

where

$$L_0=[u(y)-c_0]\frac{\partial^3}{\partial y^2\partial X}+[h_0'(y)+\beta-u''(y)]\frac{\partial}{\partial X} \tag{18}$$

$$L_1 = [u(y) - c_0]\frac{\partial^3}{\partial X^3} + \frac{\partial^3}{\partial T \partial y^2} \tag{19}$$

The solution to ψ_1 can be written:

$$\psi_1 = B(X,T)\phi_1(y,T) \tag{20}$$

according to eqs. (16)-(20), it can be seen:

$$\int_0^1 \frac{\phi_0(y,T)L_0[\psi_1]}{u(y)-c_0}\mathrm{d}y = \frac{\partial B(X,T)}{\partial X}\int_0^1 \phi_0(y,T)\frac{\partial^2 \phi_1(y,T)}{\partial y^2}\mathrm{d}y +$$

$$+\frac{\partial B(X,T)}{\partial X}\int_0^1 \frac{1}{u(y)-c_0}\phi_0(y,T)\phi_1(y,T)[h_0'(y)+\beta - u''(y)]\mathrm{d}y \tag{21}$$

$$\phi_1\Big|_{y=0,1} \to 0 \tag{22}$$

The identical equation is:

$$\phi_0 \frac{\partial^2 \phi_1}{\partial y^2} = \frac{\partial}{\partial y}\left(\phi_0 \frac{\partial \phi_1}{\partial y}\right) - \frac{\partial}{\partial y}\left(\phi_1 \frac{\partial \phi_0}{\partial y}\right) + \phi_1 \frac{\partial^2 \phi_0}{\partial y^2} \tag{23}$$

according to eqs. (15), (22) and (23):

$$\int_0^1 \phi_0(y,T)\frac{\partial^2 \phi_1(y,T)}{\partial y^2}\mathrm{d}y = \int_0^1 \phi_1(y,T)\frac{\partial^2 \phi_0(y,T)}{\partial y^2}\mathrm{d}y \tag{24}$$

substituting eqs. (14), (24) into eq. (21) yields:

$$\int_0^1 \frac{\phi_0(y,T)L_0[\psi_1]}{u(y)-c_0}\mathrm{d}y = 0 \tag{25}$$

according to eqs. (16) and (25), we have:

$$\int_0^1 \frac{\phi_0(y,T)}{u(y)-c_0}\left\{L_1[\psi_0] + J\left[\psi_0, \frac{\partial^2 \psi_0}{\partial y^2}\right] - Q_1\right\}\mathrm{d}y = 0 \tag{26}$$

In order to get the function of $A(X,T)$, to simplify the eq. (26), The final form is:

$$\frac{\partial A}{\partial T} + \chi_1(T)A\frac{\partial A}{\partial X} + \chi_2(T)\frac{\partial^3 A}{\partial X^3} = \chi_3(T)Q_2(T) \tag{27}$$

where

$$I = \int_0^1 \frac{\phi_0{}^2(y,T)}{(u(y)-c_0)^2}[h_0'(y)+\beta - u''(y)]\mathrm{d}y \tag{28}$$

$$\chi_0(T)=\frac{1}{I}\int_0^1\frac{\phi_0{}^3(y,T)}{u(y)-c_0}\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{h_0'(y)+\beta-u''(y)}{u(y)-c_0}\right]\mathrm{d}y \tag{29}$$

$$\chi_1(T)=-\frac{1}{I}\int_0^1\phi_0^2(y,T)\mathrm{d}y \tag{30}$$

$$\chi_2(T)Q_2(T)=\frac{1}{I}\int_0^1\frac{\phi_0(y,T)}{u(y)-c_0}Q_1\mathrm{d}y \tag{31}$$

Equation (27) is a variable-coefficient non-linear evolution equation with an external source, which is called VC-KdV equation with the external forcing, it can be seen from the coefficients $\chi_0(T)$, $\chi_1(T)$, the wave amplitude and velocity will be related to the topography effect.

Solutions to the VC-KdV equation

Based on the Jacobi elliptic function expansion method [1], we consider the traveling wave solution, in order to solve eq. (27), $A(X,T)$ can be expanded as the following form:

$$A(X,T)=V(X,T)+\Gamma(T) \tag{32}$$

with

$$\Gamma(T)=\int_0^T\chi_2(\tau)Q_2(\tau)\mathrm{d}\tau \tag{33}$$

Then, substituting eqs. (32) and (33) into eq. (27), yields:

$$\frac{\partial V}{\partial T}+\chi_0(T)(V+\Gamma)\frac{\partial V}{\partial X}+\chi_1(T)\frac{\partial^3 V}{\partial X^3}=0 \tag{34}$$

The form of the solution is:

$$V=V(\xi),\quad \xi=f(T)X+g(T) \tag{35}$$

where $f(T)$ and $g(T)$ are functions of time, T, in order to balance the derivative term of the highest order and the non-linear term in (34). The solution of eq. (34) is:

$$V=a_0(T)+a_1(T)\operatorname{sn}\xi+a_2(T)\operatorname{sn}^2\xi \tag{36}$$

substituting eqs. (35) and (36) into eq. (34) yields:

$$a_0{}'(T)=a_1{}'(T)=a_2{}'(T)=0$$

$$a_1(T)[f'(T)X+g'(T)]+a_1(T)f(T)\Gamma(T)\chi_0(T)+$$

$$+a_1(T)a_0(T)f(T)\chi_0(T)-a_1(T)(1+m^2)f^3(T)\chi_1(T)=0$$

$$2a_2(T)[f'(T)X+g'(T)]+2a_2(T)f(T)\Gamma(T)\chi_0(T)+$$

$$+f(T)[a_1{}^2(T)+2a_0(T)a_2(T)]\chi_0(T)-8(1+m^2)a_2(T)f^3(T)\chi_1(T)=0$$

$$f(T)a_1(T)[a_2(T)\chi_0(T)+2m^2f^2(T)\chi_1(T)]=0$$

$$f(T)a_2(T)[a_2(T)\chi_0(T)+12m^2f^2(T)\chi_1(T)]=0$$

Thus, the coefficients can be determined:

$$f(T)=k,\quad a_0(T)=C+4k^2(1+m^2)C_1,\quad a_1(T)=0,\quad a_2(T)=-12k^2m^2C_1$$

$$g(T)=-k\int_0^T\chi_0(\tau)\Gamma(\tau)\,\mathrm{d}\tau-Ck\int_0^T\chi_0(\tau)\,\mathrm{d}\tau \tag{37}$$

with the constraint $[\chi_1(\tau)]/[\chi_0(\tau)]=C_1$, where k, C, and C_1 are all none-zero constants, so the exact periodic solution of eq. (27) is:

$$A(X,T)=C+4k^2(1+m^2)C_1+\int_0^T\chi_2(\tau)Q_2(\tau)\mathrm{d}\tau-12k^2m^2C_1\,\mathrm{sn}^2\,\xi \tag{38}$$

or

$$A(X,T)=C-4k^2(2m^2-1)C_1+\int_0^T\chi_2(\tau)Q_2(\tau)\mathrm{d}\tau+12k^2m^2C_1\mathrm{cn}^2\,\xi \tag{39}$$

with

$$\xi=k\left\{X-\int_0^T[\chi_0(\gamma)\int_0^T\int_0^\tau\chi_2(\gamma)Q_2(\gamma)\mathrm{d}\gamma]\mathrm{d}\tau-C\int_0^T\chi_0(\gamma)\,\mathrm{d}\tau\right\} \tag{40}$$

that is the solitary wave solution of VC-KdV equation.

When $m\to 1$, eq. (39) can be written as:

$$A(X,T)=C-4k^2C_1+\int_0^T\chi_2(\tau)Q_2(\tau)\mathrm{d}\tau+12k^2C_1\mathrm{sech}^2\xi \tag{41}$$

$$\xi=k\left\{X-\int_0^T[\chi_0(\gamma)\int_0^\tau\chi_2(\gamma)Q_2(\gamma)\,\mathrm{d}\gamma]\mathrm{d}\tau-C\int_0^T\chi_0(\gamma)\,\mathrm{d}\tau\right\} \tag{42}$$

which is the solitary wave solution.

If we ignore the external source, that is:

$$Q_2=0 \tag{43}$$

Equation (27) degrades to:

$$\frac{\partial A}{\partial T}+\chi_0(T)A\frac{\partial A}{\partial X}+\chi_1(T)\frac{\partial^3 A}{\partial X^3}=0 \tag{44}$$

From eqs. (39) and (40), it is easy to find the solution of eq. (44):

$$A(X,T) = C + 4k^2 C_1(1+m^2) - 12k^2 m^2 C_1 \operatorname{sn}^2 \xi \tag{45}$$

with

$$\frac{\chi_1(\tau)}{\chi_0(\tau)} = C_1, \quad \xi = k\left[X - C\int_0^T \chi_0(\tau)\,\mathrm{d}\tau\right] \tag{46}$$

The degenerate solution of eq. (44) is the same with the previous study [14].

Conclusions

In the barotropic fluids, a VC-KdV equation is derived for the Rossby waves with influence of topography and external source, from the model, it is shown that the topography plays an important role in the shape of Rossby wave, including the wave amplitude, velocity and width.

We have extended the Jacobi elliptic function method with symbolic computation to non-linear equations and successfully constructed for them a series of exact solutions, some periodic-like structures and solitary wave solution for the VC-KdV equations are obtained. If we ignore external source and topography, the solution degenerates into the previous studies. What's more, the purpose of the analytical solution of the equation is to help us to explain the dynamic phenomenon in the atmosphere and ocean, so more real applications of our solutions will be discussed in our research work.

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Nomenclature

t – time co-ordinate, [s]

x, y – space co-ordinates, [m]

References

[1] Fu, Z. T., *et al.*, Equatorial Rossby Solitary Wave under the External Forcing, *Communications in Theoretical Physics*, *43* (2005), 1, pp. 45-48
[2] Tang, X. Y., *et al.*, Variable Coefficient, Non-Linear Systems Derived from an Atmospheric Dynamical System, *Chinese Physics B*, *18* (2009), 11, pp. 4622-4635
[3] Pedlosky, J., The Destabilization of Shear Flow by Topography, *Journal of Physical Oceanography*, *10* (1980), 11, pp. 1877-1879
[4] Yang, Y. Y., *et al.*, On the Generalized Eigenvalue Problem of Rossby Waves Vertical Velocity under the Condition of Zonal Mean Flow and Topography, *Applied Mathematics Letters*, *121* (2021), ID 107485
[5] Bi, Y. H., *et al.*, Research on Non-Linear Waves of Blood Flow in Arterial Vessels, *Communications in Nonlinear Science and Numerical Simulation*, *102* (2021), ID 105918
[6] Bekir, A., On Traveling Wave Solutions to Combined KdV-mKdV Equation and Modified Burgers-KdV Equation, *Communications in Nonlinear Science and Numerical Simulation*, *14* (2009), 4, pp. 1038-1042
[7] Huang, Y., *et al.*, All Exact Traveling Wave Solutions of the Combined KdV-mKdV Equation, *Advances in Difference Equations*, *2014* (2014), 1, 261

[8] Alam, M. N., Akbar, M. A., Some New Exact Traveling Wave Solutions to the Simplified MCH Equation and the (1+1)-Dimensional Combined KdV-mKdV Equations, *Journal of the Association of Arab Universities for Basic and Applied Sciences*, *17* (2015), Apr., pp. 6-13
[9] Zhang, H., *et al.*, N-Lump and Interaction Solutions of Localized Waves to the (2+1)-Dimensional Generalized KP Equation, *Results in Physics*, *25* (2021), 5, ID 104168
[10] Zhang, W. J., *et al.*, Solitary Wave, M-Lump and Localized Interaction Solutions to the (4+1)-Dimensional Fokas Equation, *Physica Scripta*, *95* (2020), 4, ID 045217
[11] Fu, Z. T., *et al.*, Structures of Equatorial Envelope Rossby Wave under the Influence of New Type of Diabatic Heating, *Chaos, Solitons and Fractals*, *22* (2004), 22, pp. 335-340
[12] Zhao, Q., *et al.*, Two-Dimensional Rossby Waves: Exact Solutions to Petviashvili Equation, *Communications in Theoretical Physics*, *45* (2006), 3, pp. 414-416
[13] Pedlosky, J., *Geophysical Fluid Dynamics second edition*, Springer, New York, USA, 1987
[14] Liu, S. K., *et al.*, Solution by Using the Jacobi Elliptic Function Expansion Method for Variable-Coefficient Non-Linear Equations, *Acta Physica Sinica-Chinese Edition*, *51* (2002), 19, pp. 1923-1926

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