LOCAL FRACTIONAL DERIVATIVE
A Powerful Tool to Model the Fractal Differential Equation

by

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In this paper, the modified Fornberg-Whitham equation is described by the local fractional derivative for the first time. The fractal complex transform and the modified reduced differential transform method are successfully adopted to solve the modified local Fornberg-Whitham equation defined on fractal sets.

Key words: *local fractional derivative, fractal complex transform, modified reduced differential transform method, modified Local Fornberg-Whitham equation, fractals*

Introduction

The fractional Fornberg-Whitham equation was first proposed for studying the qualitative behaviour of wave breaking [1]. The fractional Fornberg-Whitham equation was given:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^{2+\alpha}u}{\partial x^2\partial t^\alpha}+\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}=3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3} \tag{1}$$

with the following initial condition:

$$u(x,0)=\mathrm{e}^{1/2x} \tag{2}$$

where $\partial^\alpha/\partial t^\alpha = \mathrm{D}_x^\alpha(\cdot)$ is the Caputo fractional derivative, defined [2]:

$$\mathrm{D}_x^\alpha[f(x)]=\frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\mathrm{d}t$$

In this paper, the modified local fractional Fornberg-Whitham equation is given:

$$u_t^\alpha+u_{xxt}+u_x+u^2u_x-3u_xu_{xx}-uu_{xxx}=0 \tag{3}$$

where u_t^α is local fractional derivative, defined as [2-11]:

$$\mathrm{D}_x^{(\alpha)}f(x_0)=f^{(\alpha)}(x_0)=\left.\frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}\right|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha[f(x)-f(x_0)]}{[x-x_0]^\alpha} \tag{4}$$

where $\Delta^\alpha[f(x)-f(x0)]\cong\Gamma(1+\alpha)\Delta[f(x)-f(x_0)]$.

In our work, we will use the fractal complex transform (also called the local fractional complex transform) [12] and the modified reduced differential transform method [13] to solve the modified local fractional Fornberg-Whitham equation. The fractal complex transform can convert the local fractional differential equation into its differential partner. The modified reduced differential transform method is used to find the approximate analytical solution of the non-linear problem.

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Fractal complex transform

Consider the general form of the local fractional differential equation:

$$\begin{cases} f(u, u_t^{\alpha}, u_x^{\beta}, u_y^{\gamma}, u_z^{\lambda}, u_t^{2\alpha}, u_x^{2\beta}, u_y^{2\gamma}, u_z^{2\lambda} \ldots) = 0 \\ 0 < \alpha \le 1, \quad 0 < \beta \le 1, \quad 0 < \gamma \le 1, \quad 0 < \lambda \le 1 \end{cases} \tag{5}$$

where $u_t^{\alpha} = \partial^{\alpha} u / \partial t^{\alpha}$ is the local fractional partial derivative, defined as [2-11]:

$$\frac{\partial^{\alpha} u(x, y, z, t)}{\partial t^{\alpha}} = \lim_{t \to t_0} \frac{\Delta^{\alpha}[f(x, y, z, t_0) - f(x_0, y, z, t)]}{(t_0 - t)^{\alpha}}$$

with

$$\Delta^{\alpha}[f(x, y, z, t_0) - f(x_0, y, z, t)] \cong \Gamma(1+\alpha)\Delta[f(x, y, z, t_0) - f(x_0, y, z, t)]$$

and u is continuous (but not necessarily differentiable) function.

The fractal complex transform reads:

$$T = \frac{pt^{\alpha}}{\Gamma(1+\alpha)}, \quad X = \frac{qx^{\beta}}{\Gamma(1+\beta)}, \quad Y = \frac{ky^{\gamma}}{\Gamma(1+\gamma)}, \quad Z = \frac{lz^{\lambda}}{\Gamma(1+\lambda)}$$

where p, q, k, and l are unknown constants.

Making use of the basic properties of the local fractional derivative and above transform, we have the following [12]:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = p \frac{\partial u}{\partial T}, \quad \frac{\partial^{\beta} u}{\partial x^{\beta}} = q \frac{\partial u}{\partial X}, \quad \frac{\partial^{\gamma} u}{\partial y^{\gamma}} = k \frac{\partial u}{\partial Y}, \quad \frac{\partial^{\lambda} u}{\partial z^{\lambda}} = k \frac{\partial u}{\partial Z}$$

Therefore, we can easily convert the local fractional PDE into partial differential equations in the classical sense.

The modified reduced differential transform method

In this section, the basic definition of the modified reduced differential transform method is introduced [13].

Definition 1. The new reduced differential transform of $u(x, t)$ at $t = t_0$ is represented [13]:

$$U_k(x) = \frac{1}{\Gamma(k\alpha + 1)} \left[\frac{\partial^k}{\partial t^k} u(x, t) \right]_{t=t_0} \tag{6}$$

where α is a parameter which describes the order of time-fractional derivative.

Definition 2. The differential inverse transform of $U_k(x)$ is represented [13]:

$$u(x, t) = \sum_{k=0}^{\infty} U_k(x)(t - t_0)^{k\alpha} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} \left[\frac{\partial^k}{\partial t^k} u(x, t) \right]_{t=t_0} t^{k\alpha} \tag{7}$$

From eqs. (6) and (7), the following theorems can be obtain, see [13].

Theorem 1. If:

$$w(x, t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}} u(x, t)$$

then [13]:

$$\mathrm{MRDT}[w(x, t)] = \frac{\Gamma(k\alpha + N\alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+N}(x)$$

Theorem 2. If $w(x, t) = [u(x, t)]^k$, then [13]:

$$\mathrm{MRDT}[w(x,t)] = W_k(x) = \begin{cases} U_0(x), k=0 \\ \displaystyle\sum_{n=1}^{k} \frac{(m+1)n-k}{kU_0(x)} U_n(x) W_{k-n}(x), k \geq 1 \end{cases}$$

Numerical example

We now consider the modified local fractional Fornberg-Whitham equation:

$$u_t^\alpha + u_{xxt} + u_x + u^2 u_x - 3u_x u_{xx} - uu_{xxx} = 0 \tag{8}$$

with the initial condition：

$$u(x,0) = \frac{3}{4}\left(\sqrt{15}-5\right)\mathrm{sech}^2(cx) \tag{9}$$

where

$$c = \frac{1}{20}\sqrt{10\left(5-\sqrt{15}\right)}$$

To solve eq. (8), we convert it into the differential partner by the fractal complex transform:

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)} \tag{10}$$

We can easily convert eq. (8) into the classical non-linear PDE:

$$u_T + u_{xx} + u_x + u^2 u_x = 3u_x u_{xx} + uu_{xxx} \tag{11}$$

Making use of the modified reduced differential transform method, we can obtain that:

$$(k+1)U_{k+1}(x) - (k+1)\frac{\partial^2}{\partial x^2}U_{k+1}(x) = -\frac{\partial}{\partial x}U_k(x) - \sum_{r=0}^{k}\sum_{s=0}^{r} U_{k-r}(x)U_{r-s}(x)\frac{\partial}{\partial x}U_s(x) +$$
$$+\sum_{r=0}^{k} U_{k-r}(x)\frac{\partial^3}{\partial x^3}U_r(x) + 3\sum_{r=0}^{k} U_{k-r}(x)\frac{\partial^2}{\partial x^2}U_r(x)$$

From the initial condition eq. (9), we have that:

$$U_0(x) = \frac{3}{4}\left(\sqrt{15}-5\right)\mathrm{sech}^2(cx)$$

In this case, we can obtain the followings:

$$U_1(x) = -\frac{105}{8}x + \frac{27\sqrt{15}}{8}x + \frac{31}{8}x^3 - \sqrt{15}x^3 \ldots \quad U_2(x) = \frac{465}{8} - 15\sqrt{15} - \frac{825}{16}x^2 + \frac{213\sqrt{15}}{16}x^2 \ldots$$
$$U_3(x) = 305x - \frac{315\sqrt{15}}{4}x - \frac{36805}{192}x^3 + \frac{9503}{64}\sqrt{\frac{5}{3}}x^3 \ldots$$

So, we have the solution of eq. (11):

$$u(x,T) = U_0(x)T^0 + U_1(x)T^1 + U_2(x)T^2 + U_3(x)T^3 + \ldots = \sum_{k=0}^{\infty} U_k(x)T^k$$

Substituting eq. (10) into the previous results, we obtain the approximate solution of eq. (8):

$$u(x,t) = U_0(x)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^0 + U_1(x)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^1 + U_2(x)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^2 + \ldots \tag{12}$$

Finally, we get the exact solution of eq. (8):

$$u(x,t)=U_0(x)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^0+U_1(x)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^1+U_2(x)\left[\frac{t^\alpha}{\Gamma(1+\alpha)}\right]^2+\ldots \tag{13}$$

For $\alpha = 1$, eq. (12) can be written:

$$u(x,t)=\frac{3}{4}\left(\sqrt{15}-5\right)\operatorname{sech}^2\left\{c\left[x-\left(5-\sqrt{15}\right)t\right]\right\} \tag{14}$$

Conclusion

In the present work, the modified local fractional Fornberg-Whitham equation containing the local fractional derivative was described for the first time. We used the fractal complex transform and modified reduced differential transform method to determine the approximate analytical solution of the modified local fractional Fornberg-Whitham equation. It is illustrated that the proposed method is more reliable, efficient and accurate.

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