OPTIMUM SOLUTIONS OF SPACE FRACTIONAL ORDER DIFFUSION EQUATION

by

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The present paper is concerned with the implementation of optimal homotopy asymptotic method to handle the approximate analytical solutions of fractional order partial differential equations. Fractional differential equations have great importance regarding distinct fields of science and engineering. Approximate solutions of space fractional order diffusion model and its various special cases are handled using the innovative proposed method. The space fractional derivatives are described in the Caputo sense. The results obtained by the proposed method are compared with various methods. The proposed method demonstrates excellent accuracy and strength over various methods.

Key words: *optimal homotopy asymptotic method, analytical solutions, space fractional order diffusion equation*

Introduction

The applications of fractional calculus are supposed to describe numerous phenomenon in applied sciences, such as anomalous diffusion transport, fluid-flow in porous materials, dynamics in self-similar structures, acoustic wave propagation in viscoelastic materials, financial theory, signal processing, electric conductance of biological systems and others have been studied, see [1-14].

Many researchers have calculated the analytical solution of fractional differential equations using certain procedures like the Laplace transform, mellin transform, Fourier transform, and *etc.* [8, 15]. These are those exact analytical solutions of only a few easy cases and equivalent to a few functions such as the Fox H-function and the hyperbolic geometric function [8, 15]. The reason of difficulties in finding exact solutions for most problems and the complexity of computing these special functions limits the applications of applied fractional differential equations in engineering and scientific computing fields. Many researchers have developed numerical algorithms [16-18] for solving the fractional differential equations, including the finite difference method, finite element method, and spectral element method to achieve the goal.

We consider the space fractional order diffusion equation [1-6]:

$$\frac{\partial z\left(x,t\right)}{\partial t} = p(x,t)\frac{\partial^{\alpha} z(x,t)}{\partial x^{\alpha}} + q(x,t), \quad 0 \le x \le 1, \quad t \in [0,T], \quad 1 < \alpha \le 2 \tag{1}$$

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with initial and boundary conditions:

$$z(x,0)=f(x),\ \ 0<x<1,\ \ z(0,t)=g_1(t),\ \ 0<t\le T,\ \ z(1,t)=g_2(t),\ \ 0<t\le T \tag{2}$$

where the fractional derivative is defined in Caputo sense and the function $q(x, t)$ is the source term and $p(x, t)$ is diffusion coefficient (or diffusivity). When $\alpha=2$, the proposed class of eqs. (1) and (2) of space fractional order diffusion equation becomes classical second order diffusion equation. Fractional order diffusion equations are generalizations of classical diffusion equations. These equations play important roles in modeling anomalous diffusion and sub-diffusion systems, description of a fractional random walk, unification of diffusion and wave propagation phenomena, see [19], and the references therein. In the analysis of the numerical method that follows, we will assume that eqs. (1) and (2) has a unique and sufficiently smooth solution. The case of $1<\alpha<2$ models a super-diffusive flow in which a cloud of diffusing particles spreads at a faster rate than the classical diffusion model predicts [20, 21], and $\alpha=1$ corresponds to the classical advective flow. Natural types of the solution of the eqs. (1) and (2) were observed and sought by numerous methods [1-6]. The main focus of this article is to extend the optimal homotopy asymptotic method (OHAM) to obtain approximate analytical solution of the eqs. (1) and (2) including a few particular cases.

Mathematical procedure of the OHAM

In this section, we derive OHAM for fractional model eq. (l). For this, we may consider the fractional model eq. (l):

$$B[z(x,t)]+f(x,t)=0,\ \ x\in\Omega,\ \ t\ge 0 \tag{3}$$

here the symbol B is differential operator that may be integer order derivative operator, time or space fractional order derivative operator. The Ω is the domain, $z(x,t)$ – the solution of eq. (3), x and t – the spatial and time variables, respectively, and $f(x,t)$ – a known function. About the OHAM procedure, we can formulate the operator B in the term of L and N differential operators used in eq. (3):

$$L[z(x,t)]+N[z(x,t)]+f(x,t)=0,\ \ x\in\Omega$$

where L is the simpler one differential operator for eq. (3) that could be the simple part: $\partial^\alpha z(x,t)/\partial x^\alpha$ of eq. (3) such that it could be easily solvable by using the inverse operator j^α (Riemann-Liouville integral operator [1-6]), whereas N is the differential operator of eq. (3) that would be the rest part of the differential eq. (3). Now, let us consider $z_0(x,t):\Omega\to\Re$ is the solution of:

$$\begin{gathered} L[z_0(x,t)]+f(x,t)=0 \\ \varphi\left[z_0(x,t),\frac{\partial z_0(x,t)}{\partial t}\right]=0 \end{gathered} \tag{4}$$

that is also continuous function. eq. (3) has solution $z(x,t):\Omega\to\Re$, which will be also continuous function. Then we define the homotopy $\Psi(x,t;m):\Omega\times[0,1]\to\Re$ which satisfies:

$$(1-m)\{L[\Psi(x,t;m)]+f(x,t)\}=H(x,m)\{B[\Psi(x,t;m)]+f(x,t)\} \tag{5}$$

where (x,t,m) belongs to the domain of $\Psi(x,t;m)$, $m\in[0, 1]$ – the embeding parameter, $H(x,m)$ – the arbitrary chosen auxiliary function for eq. (3). The convergence of OHAM solution to the exact solution of eq. (3) depends on auxiliary function $H(x,m)$. Basically the shape of auxiliary function follows the terms appearing in non-linear part N such that the product of auxiliary function and N be of the same shape. It is necessary that $H(x,m)$ must not equal to zero for all m except at $m=0$. Moreover, for $m=0$, eq. (5) will be transferred into eq. (4) and for $m=1$. Equation (5) will be transferred into eq. (3).

Per the defined homotopy:

$$\Psi(x,t;m) = z_0(x,t)\,, \text{ at } m=0$$

$$\Psi(x,t;m) = z(x,t)\,, \text{ at } m=1.$$

As a result, when m varies from 0 to 1, $\Psi(x,t;m)$ deforms from $z_0(x,t)$ into $z(x,t)$. But $z_0(x,t)$ is the solution of differential eq. (4), which is obtained from eq. (5) and eq. (3) at $m=0$. Here, keeping in mind that product of auxiliary function and N will have same shape, we are going to choose auxiliary function $H(x,m)$ for fractional model eq. (1):

$$H(x,m) = C_1 m + C_2 m^2 + C_3 m^3 + \ldots\ldots.. + C_k m^k + \ldots.,$$

where $C_1, C_2, C_3, \ldots., C_k, \ldots$ are invariant and these are to be sought later.

We expand $\Psi(x,t;m,C_1,C_2\ldots)$ in Taylor's expansion with respect to m so that:

$$\Psi(x,t;m,C_1,C_2,\ldots) = z_0(x,t) + \sum_{r=1}^{\infty} z_r(x,t;C_1,C_2\ldots C_r)m^r \tag{6}$$

We substitute eq. (6) into eq. (5) and equate coefficient of the same powers of m. In addition, we obtain zeroth-order linear differential equation which is given in eq. (4) and the succeeded two simple differential equations:

$$L\left[z_1(x,t)\right] = C_1 N_0\left[z_0(x,t)\right], \quad \varphi\left[z_1(x,t), \frac{\partial z_1(x,t)}{\partial t}\right] = 0$$

$$L\left[z_2(x,t)\right] = C_2 N_0\left[z_0(x,t)\right] + C_1 N_1\left[z_0(x,t), z_1(x,t)\right] + (1+C_1)L\left[z_1(x,t)\right]$$

$$\varphi\left[z_2(x,t), \frac{\partial z_2(x,t)}{\partial t}\right] = 0$$

respectively. Furthermore, the general governing r^{th}-order problem of the analytical solution $z_r(x,t)$:

$$L\left[z_r(x,t)\right] = L\left[z_{r-1}(x,t)\right] + C_r N_0\left[z_0(x,t)\right] + \sum_{j=1}^{r-1} C_j \begin{Bmatrix} L\left[z_{r-j}(x,t)\right] + \\ +N_{r-j}\left[z_0(x,t), z_1(x,t), \ldots, z_{r-j}(x,t)\right] \end{Bmatrix}$$

$$\varphi\left[z_r(x,t), \frac{\partial z_r(x,t)}{\partial t}\right] = 0, \; r = 2, 3, \ldots,$$

where $N_{r-j}[z_0(x,t), z_1(x,t), \ldots, z_{r-j}(x,t)]$, $j = 1, 2\ldots, r$ in the general governing r^{th}-order differential problem is coefficient of the m^{r-j} with respect to the embed parameter m and:

$$N\left[\Psi(x,t;m,C_1,C_2,\ldots)\right] = N_0\left[z_0(x,t)\right] + \sum_{i=1}^{\infty} N_i(z_0, z_1, \ldots\ldots, z_i)m^i$$

It is evident and can be observed that the solutions $z_i(x,t)$, $i \geq 0$ are governed by the series problems, which are constructed by using $\partial^\alpha z(x,t)/\partial x^\alpha$ and non-linear part of the eq. (3). Now, because of the previous defined homotopy, the series of eq. (6) converges at $m=1$:

$$\tilde{z}(x,t;C_1,C_2,...) = z_0(x,t) + \sum_{i\geq 1} z_i(x,t;C_1,C_2,...,C_i)$$

It can be measured approximately:

$$\tilde{z}(x,t;C_1,C_2,...) = z_0(x,t) + \sum_{i=1}^{n} z_i(x,t;C_1,C_2,...,C_i) \tag{7}$$

We substitute eq. (7) into eq. (3) to obtain the residual so that:

$$R(x,t;C_1,C_2,..,) = B\left[\tilde{z}(x,t;C_1,C_2,...)\right] + f(x,t) \tag{8}$$

Least square method for to get value of the constants $C_1, C_2, C_3, \ldots$, one can exercise:

$$j(C_1,C_2,C_3,...) = \int_0^t \int_\Omega R^2(x,t;C_1,C_2,...)\mathrm{d}x\mathrm{d}t$$

and value of the auxiliary constants $C_1, C_2, C_3, ...$ could be obtained by solving the system of algebraic equation in $C_1, C_2, C_3, ...$:

$$\frac{\partial j}{\partial C_1} = \frac{\partial j}{\partial C_2} = \frac{\partial j}{\partial C_3} = ... = 0$$

It can be observed during implementation of OHAM that $C_1, C_2...$, for the OHAM solution "$\tilde{z}(x,t;C_1,C_2,...)$" of the eq. (3) is well fit. The value of the constants C_1, C_2, C_3, ..., C_n may be sought by another procedure defined in [22].

Numerical simulations

In this portion, we apply OHAM for the four cases of space fractional order diffusion eq. (1). For this, we must transfer each original problem to simplest problems series. Each simple problem well be solving by apply the Riemann-Liouville well know operator j^α, for basic definitions see [1-6] on both sides.

Example 1. Let us consider a version of fractional order diffusion equation eq. (1), see [1-6]:

$$\frac{\partial z(x,t)}{\partial t} = p(x,t)\frac{\partial^\alpha z(x,t)}{\partial x^\alpha} + q(x,t), \quad 0 \leq x \leq 1, \quad t \in [0,T], \quad \alpha = 1.8 \tag{9}$$

initial and boundary conditions:

$$\begin{aligned} &z(x,\,0) = x^2 - x^3, && 0 < x < 1 \\ &z(0,\,t) = 0, && 0 < t \leq T \\ &z(1,\,t) = 0, && 0 < t \leq T \end{aligned}$$

The diffusion coefficient: $p(x,\,t) = \Gamma(1.2)x^{1.8}$ and source term $q(x,t) = (6x^3 - 3x^2)\mathrm{e}^{-t}$.
Note that the exact solution of diffusion eq. (9) [1-6] is $z(x,t) = (x^2 - x^3)\mathrm{e}^{-t}$.
In same way, here, we are opting differential operators L and N for eq. (9):

$$L\left[\Psi(x,t;m)\right]=\frac{\partial^{1.8}\Psi(x,t;m)}{\partial x^{1.8}},\quad N\left[\Psi(x,t;m)\right]=-\frac{1}{p(x,t)}\frac{\partial\Psi(x,t;m)}{\partial t},\quad f(x,t)=\frac{q(x,t)}{p(x,t)}$$

where $m\in[0,\ 1]$ is embedded parameter provided in deformation equation and $q(x,t)$ is source term.

Initial condition: $\Psi\left(0,\ t;m\right)=0$.

Following the fundamental concept of OHAM, we initiate with:

– zeroth-order problem

$$\frac{\partial^{1.8}z_0(x,t)}{\partial x^{1.8}}=0,\ z_0(0,t)=0$$

Its solution is:

$$z_0(x,t)=\left[-1.2x^3+1.5x^2\right]e^{-t}. \tag{10}$$

– first-order problem

$$\frac{\partial^{1.8}z_1(x,t)}{\partial x^{1.8}}=C_1N_0\left[z_0(x,t)\right],\ z_1(0,t)=0$$

Its solution is:

$$z_1(x,t)=C_1\left[-0.24x^3+0.7498x^2\right]e^{-t}. \tag{11}$$

– second-order problem

$$\frac{\partial^{1.8}z_2(x,t)}{\partial x^{1.8}}=C_2N_0\left[z_0(x,t)\right]+C_1N_1\left[z_0(x,t),z_1(x,t)\right]+(1+C_1)L_1\left[z_1(x,t)\right],\ z_2(x,\ 0)=0$$

Its solution is:

$$z_2(x,t)=\begin{bmatrix}(C_1^{\,2}+C_1)(-0.24x^3+0.7498x^2)+0.7498C_2x^2-\\-0.24C_2x^3+0.375C_1^{\,2}x^2-0.048C_1^{\,2}x^3\end{bmatrix}e^{-t} \tag{12}$$

– third-order problem

$$\frac{\partial^{1.8}z_3(x,t)}{\partial x^{1.8}}=C_3N_0\left[z_0(x,t)\right]+C_2\left\{N_2\left[z_0(x,t),z_1(x,t),z_2(x,t)\right]+L_1\left[z_1(x,t)\right]\right\}+C_1N_0(z_0,z_1)+$$
$$+\left(1+C_1\right)L_2\left[z_2(x,t)\right],\ z_3(0,t)=0$$

Its solution is:

$$z_3(x,t)=$$
$$\begin{bmatrix}(C_1+1)[C_1(C_1+1)(-0.24x^3+0.7498x^2)+0.7498C_2x^2-0.24C_2x^3+0.375C_1^{\,2}x^2-\\-0.048C_1^{\,2}x^3]+0.75C_3x^2-0.24C_3x^3+C_1C_2(-0.24x^3+0.7498x^2)+0.375C_1^{\,2}x^2-\\-0.048C_1^{\,2}x^3+0.56\ C_1^{\,3}x^2-0.0576C_1^{\,3}x^3+0.7498C_1C_2x^2-0.096C_1C_2x^3\end{bmatrix}e^{-t} \tag{13}$$

In this way, the higher order equations can be obtained and solved. We take only the first three solutions and add eqs. (10)-(13):

$$\tilde{z}(x,t) = z_0(x,t) + z_1(x,t) + z_2(x,t) + z_3(x,t)$$

$$\tilde{z}(x,t) =$$

$$= \begin{bmatrix} (C_1+1)\{C_1(C_1+1)(-0.24x^3+0.7498x^2)+0.7498C_2x^2-0.24C_2x^3+0.375C_1^2x^2- \\ -0.048C_1^2x^3\}+1.5x^2-1.2x^3+C_1(-0.24x^3+0.7498x^2)+C_1(C_1+1)(-0.24x^3+ \\ +0.7498x^2)+0.7498C_2x^2-0.24C_2x^3)+0.75C_3x^2-0.24C_3x^3+C_1C_2(-0.24x^3+ \\ +0.7498x^2)+0.7498\ C_1^2x^2-0.096\ C_1^2x^3+0.562\ C_1^3x^2-0.0576C_1^3x^3+ \\ +0.7498C_1C_2x^2-0.096C_1C_2x^3 \end{bmatrix} \mathrm{e}^{-t} \tag{14}$$

Next, we computed residual R for obtaining the values of auxiliary constants C_1, C_2 and C_3 via using least square method. Thus, from the residual we computed:

$$C_1 = 0.1125947253, \quad C_2 = 2.3115573170, \quad \text{and} \quad C_3 = -6.466882288$$

We are substituting the values of auxiliary constants into eq. (14), then we obtained third order approximate OHAM solution of *Example 1*:

$$\tilde{z}(x,t) = -0.9999983\,\mathrm{e}^{-1.0t}x^3 + 1.000001\,\mathrm{e}^{-1.0t}x^2$$

Tables 1 and 2 show the exact solution, OHAM approximation and absolute errors for the *Example 1*, whereas absolute error has been measured by $L_\infty = |z_{\text{exact}}(x,t) - z_{\text{OHAM}}(x,t)|$. The OHAM and exact solutions are plotted for different values of x and t in figs. 1 and 2 and curves of both OHAM and exact solution are exactly matching.

Table 1. Various numerical results of *Example 1* by OHAM at $t = 2$

x	Exact	OHAM	L_∞ [1] at $m = 3$	L_∞ [2]	L_∞ [3]	L_∞ [4]	L_∞ [5]	L_∞ (OHAM)
0.0	0.000000000	0.000000000	0.00 e–7	0.300 e–5	0.00	0.00 e–6	2.726496 e–4	0.00
0.1	0.001218018	0.001218020	3.77 e–7	0.418 e–5	4.47 e–6	3.33 e–6	3.455890 e–4	2.22 e–9
0.2	0.004330729	0.004330739	6.25 e–7	0.545 e–5	2.78 e–7	5.65 e–6	3.809670 e–4	9.78 e–9
0.3	0.008526123	0.008526147	7.59 e–7	0.618 e–5	5.81 e–6	7.05 e–6	3.809103 e–4	2.41 e–8
0.4	0.012992190	0.012992230	7.97 e–7	0.649 e–5	1.02 e–5	7.64 e–6	3.514280 e–4	4.65 e–8
0.5	0.016916910	0.016916990	7.58 e–7	0.640 e–5	1.17 e–5	7.52 e–6	3.009263 e–4	7.83 e–8
0.6	0.019488280	0.019488400	6.58 e–7	0.595 e–5	1.08 e–5	6.80 e–6	2.387121 e–4	1.21 e–7
0.7	0.019894290	0.019894460	5.14 e–7	0.532 e–5	8.54 e–6	5.59 e–6	1.735125 e–4	1.76 e–7
0.8	0.017322920	0.017323160	3.45 e–7	0.460 e–5	6.06 e–6	3.98 e–6	1.119821 e–4	2.45 e–7
0.9	0.010962160	0.010962490	1.68 e–7	0.379 e–5	3.67 e–6	2.08 e–6	0.572150 e–4	3.28 e–7
1.0	0.000000000	0.000000428	0.00 e–7	0.300 e–5	0.00 e–6	0.00 e–6	0.072566 e–4	4.28 e–7

Example 2. Consider the model [1, 4, 6], a case of eq. (1):

$$\frac{\partial z(x,\, t)}{\partial t} = p(x,t)\frac{\partial^\alpha z(x,\, t)}{\partial x^\alpha} + q(x,t), \;\; 0 \le x \le 1, \;\; t \in [0,T], \; \alpha = 1.8, \tag{15}$$

Initial and boundary conditions are:

$$z(x,\,0) = x^3, \qquad 0 < x < 1$$

$$z(0,\,t) = 0, \qquad 0 < t \le T$$

Table 2. Various numerical results of *Example 1* by OHAM at $t = 1$

x	Exact	OHAM	L_∞ [4]	L_∞ (OHAM)
0.0	0.00000000	0.000000000	0.00 e–6	0.00
0.1	0.00331092	0.003310921	5.46 e–6	6.02 e–9
0.2	0.01177214	0.011772170	8.51 e–6	2.66 e–8
0.3	0.02317640	0.023176470	9.60 e–6	6.54 e–8
0.4	0.03531643	0.035316550	9.18 e–6	1.26 e–7
0.5	0.04598493	0.045985140	7.69 e–6	2.13 e–7
0.6	0.05297464	0.052974970	5.60 e–6	3.29 e–7
0.7	0.05407828	0.054078760	3.33 e–6	4.79 e–7
0.8	0.04708857	0.047089230	1.34 e–6	6.65 e–7
0.9	0.02979823	0.029799130	8.39 e–8	8.92 e–7
1.0	0.00000000	0.000001164	0.00	1.16 e–6

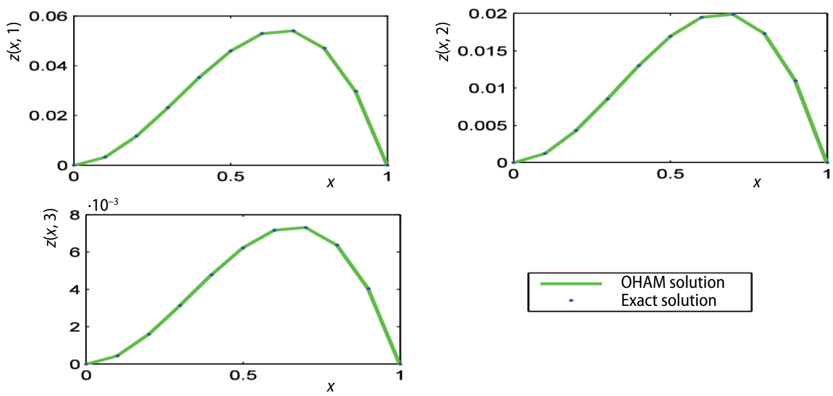

Figure 1. The OHAM solution and exact solution of *Example 1* for different values of *x* and *t* (*for color image see journal web site*)

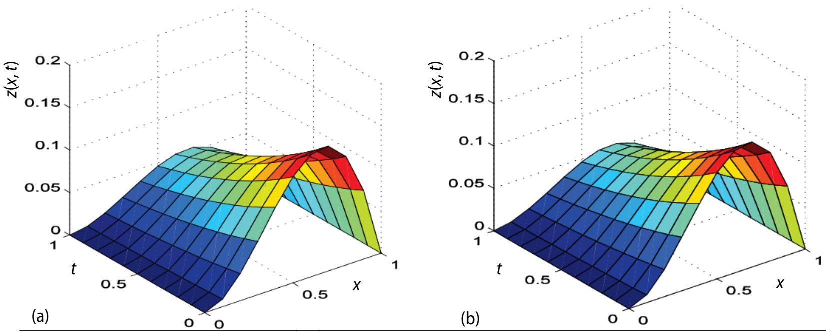

Figure 2. The OHAM solution and exact solution of *Example 1* for different values of *x* and *t*; (a) OHAM solution, (b) exact solution (*for color image see journal web site*)

$$z(1, t) = e^{-t}, \qquad 0 < t \le T$$

The diffusion coefficient: $p(x,t) = [\Gamma(2.2)/6\,]x^{2.8}$ and source term: $q(x,t) = -(1+x)x^3\mathrm{e}^{-t}$. Note that the exact solution of diffusion eq. (15) [1, 4, 6] is $z(x,t) = x^3\mathrm{e}^{-t}$.

We apply the same procedure as applied for *Example 1*, we obtained third order approximate OHAM solution of *Example 2* as follow:

$$\tilde{z} = 0.935\,\mathrm{e}^{-t}x^3 + 0.00756\,\mathrm{e}^{-t}x^2 - 7.63e^{-5}\mathrm{e}^{-t}x + 6.89\mathrm{e}^{-8}\mathrm{e}^{-t}$$

Table 3. Various numerical results of *Example 2* by OHAM at t = 1

x	Exact	OHAM	L_∞ [6]	L_∞ (OHAM)
0.00	0.0	2.5 e–8	2.8275 e–5	2.5 e–8
0.04	2.4 e–5	2.5 e–5		1.8 e–6
0.08	1.9 e–4	1.9 e–4		3.4 e–6
0.12	6.4 e–4	6.3 e–4		4.4 e–6
0.16	1.5 e–3	1.5 e–3		3.1 e–5
0.20	2.9 e–3	2.9 e–3		8.5 e–5
0.24	5.1 e–3	4.9 e–3		1.8 e–4
0.28	8.1 e–3	7.8 e–3		3.1 e–4

Table 3 shows the exact solution, OHAM approximation and absolute errors of the *Example 2*. The OHAM solution and exact solution are plotted for different values of x and t in fig. 3 and curves of both OHAM and exact solution are very well matching.

Example 3. Consider the fractional model [3] as a special case of eq. (1):

Note that the exact solution [3] is $z(x,t) = x^2 e^{-t}$.

We apply the same procedure as applied for *Examples 1 and 2*, we obtained third order approximate OHAM solution of *Example 3*: $\tilde{z}(x,t) = 1.0000001x^2 e^{-1.0t}$.

Table 4 shows the exact solution, OHAM approximation and absolute errors of the *Example 3*. The OHAM solution and exact solution are plotted for different values of x and t in figs. 4 and 5 and curves of both OHAM and exact solution are very well matching.

Table 4. Various numerical results of *Example 3* by OHAM

	t = 1			t = 2		
x	Exact	OHAM	L_∞	Exact	OHAM	L_∞
0.00	0.00000000000	0.00000000000	0.00	0.00000000000	0.00000000000	0.00
0.05	0.00091969860	0.00091969869	8.50e–11	0.00033833821	0.00033833824	3.13e–11
0.10	0.00367879441	0.00367879475	3.40e–10	0.00135335283	0.00135335296	1.25e–10
0.15	0.00827728743	0.00827728819	7.65e–10	0.00304504387	0.00304504415	2.82e–10
0.20	0.01471517765	0.01471517901	1.36e–9	0.00541341133	0.00541341183	5.01e–10
0.25	0.02299246507	0.02299246720	2.13e–9	0.00845845520	0.00845845598	7.82e–10
0.30	0.03310914971	0.03310915277	3.06e–9	0.01218017549	0.01218017662	1.13e–9
0.35	0.04506523154	0.04506523571	4.17e–9	0.01657857220	0.01657857373	1.53e–9
0.40	0.05886071059	0.05886071603	5.44e–9	0.02165364532	0.02165364732	2.00e–9
0.45	0.07449558684	0.07449559373	6.89e–9	0.02740539486	0.02740539739	2.53e–9
0.50	0.09196986029	0.09196986880	8.50e–9	0.03383382081	0.03383382394	3.13e–9
0.55	0.11128353100	0.11128354120	1.03e–8	0.04093892318	0.04093892696	3.79e–9
0.60	0.13243659880	0.13243661110	1.22e–8	0.04872070197	0.04872070647	4.51e–9
0.65	0.15542906390	0.15542907830	1.44e–8	0.05717915717	0.05717916245	5.29e–9
0.70	0.18026092620	0.18026094280	1.67e–8	0.06631428879	0.06631429492	6.13e–9
0.75	0.20693218570	0.20693220480	1.91e–8	0.07612609682	0.07612610386	7.04e–9
0.80	0.23544284230	0.23544286410	2.18e–8	0.08661458127	0.08661458928	8.01e–9
0.85	0.26579289620	0.26579292080	2.46e–8	0.09777974214	0.09777975118	9.04e–9
0.90	0.29798234730	0.29798237490	2.76e–8	0.10962157940	0.10962158960	1.01e–8
0.95	0.3320111957	0.33201122640	3.07e–8	0.12214009310	0.12214010440	1.13e–8
1.00	0.3678794412	0.36787947520	3.40e–8	0.13533528320	0.13533529580	1.25e–8

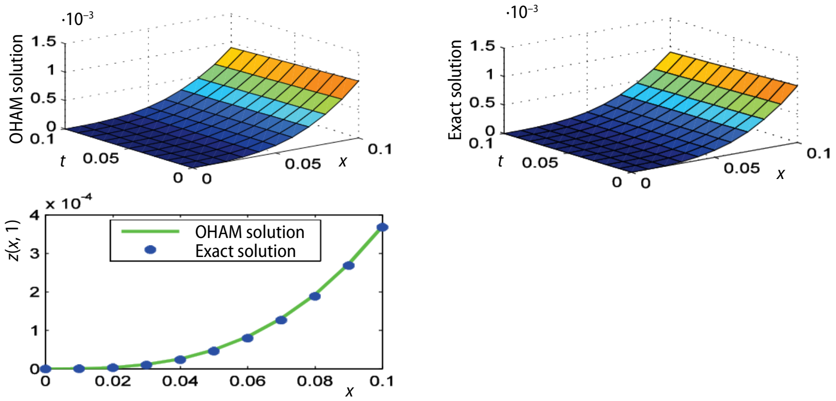

Figure 3. The OHAM solution and exact solution of *Example 2* for different values of *x* and *t* (*for color image see journal web site*)

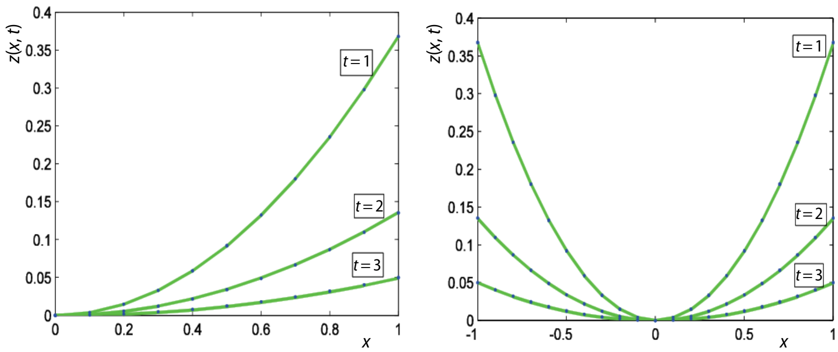

Figure 4. The OHAM solution and exact solution of *Example 3* for different values of *x* and *t*; blue color denotes exact and green color denotes OHAM
(*for color image see journal web site*)

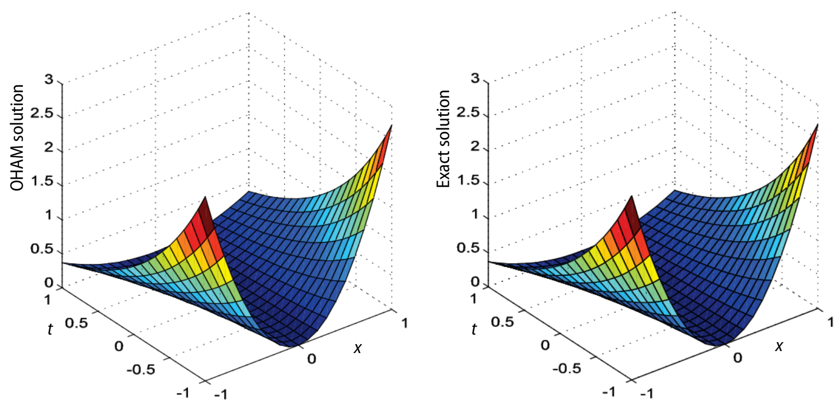

Figure 5. The OHAM solution and exact solution of *Example 3* for different values of *x* and *t* (*for color image see journal web site*)

Conclusion

In this work, a recently developed analytical technique OHAM has been employed to find approximate analytical solution of fractional order partial diffusion equation. Like other methods, OHAM does not require discretization of domain under consideration. In the recent emergence of fractional order partial differential models in some fields of applied mathematics, it is necessary to examine the methods for the solutions of such models and we anticipate that this work is a step towards the solutions of fractional problems. It has been observed that approximate solutions by extended formulation are in excellent agreement with the exact solutions. It is evidenced from simulation section that the OHAM overcomes the methods [1-6]. This is a significant progress in computing solution of space-fractional PDE. The application of OHAM into solving every kind of time-fractional and space-fractional differential equations will be our further consideration.

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References

[1] Sweilam, N. H., *et al*., On the Numerical Solution of Space Fractional Order Diffusion Equation via Shifted Chebyshev Polynomials of the Third Kind, *J. of K. Saud Uni.–Sci., 28* (2016), 1, pp. 41-47
[2] Khader, M. M., On the Numerical Solutions for the Fractional Diffusion Equation, *Commun. Nonli. Sci. Numer. Simul., 16* (2011), 6, pp. 2535-2542
[3] Saadatmandi, A., Dehghan, M., A Tau Approach for Solution of the Space Fractional Diffusion Equation, *Comput. and Math. with Appl., 62* (2011), 3, pp. 1135-1142
[4] Sweilam, N. H., *et al*., Second Kind Shifted Chebyshev Polynomials for Solving Space Fractional Order Diffusion Equation, *Chaos, Solitons & Fractals, 73* (2015), Apr., pp. 141-147
[5] Khader, M. M., *et al.*, An Efficient Numerical Method for Solving the Fractional Diffusion Equation, *J. of Appl. Math. & Bioinfor., 1* (2011), 2, pp. 1-12
[6] Tadjeran, C., *et al.*, A Second-Order Accurate Numerical Approximation for the Fractional Diffusion Equation, *J. of Comput. Phy., 213* (2006), 1, pp. 205-213
[7] Oldham, K. B., Spanier, J., *The Fractional Calculus*, Academic Press, New York, USA, 1974
[8] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, Cal., USA, 1999
[9] Kilbas, A., *et al.*, *Theory and Applications of Fractional Differential Equations*, Elsevier, Boston, Mass., USA, 2006
[10] Mainardi, F., *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010
[11] Anastassiou, G.-A., *Advances on Fractional Inequalities*, Springer, New York, USA, 2011.
[12] Khan, H., *et al.*, Existence Theorems and Hyers-Ulam Stability for a Coupled System of Fractional Differential Equations with p-Laplacian Operator, *B. Value Probl., 157* (2017), Dec., pp. 1-16
[13] Khan, H., *et al.*, On Iterative Solutions and Error Estimations of a Coupled System of Fractional Order Differential-Integral Equations with Initial and Boundary Conditions, *Differ. Equ. Dyn. Syst.,* 2017. DOI: 10.1007/s12591-017-0365-7
[14] Jafari, H., *et al.*, Existence Criterion for the Solutions of Fractional Order p-Laplacian Boundary Value Problems, *Boundary Value Problems, 164* (2015), Dec., pp. 1-10
[15] Mainardi, F., *et al.*, Fox H Functions in Fractional Diffusion, *J. Comput. Appl. Math.,178* (2005), 1-2, pp. 321-331
[16] Chen, C., *et al.*, Numerical Methods for Solving a Two-Dimensional Variable-Order Anomaloussub Diffusion Equation, *Math. Comp., 81* (2012), pp. 345-366
[17] Su, L., *et al.*, A Characteristic Difference Method for the Transient Fractional Convection-Diffusion Equations, *Appl. Numer. Math., 61* (2011), 8, pp. 946-960
[18] Langlands, T., Henry, B., The Accuracy and Stability of an Implicit Solution Method for the Fractional Diffusion Equation, *J. Comput. Phys., 205* (2005), 2, pp. 719-736

[19] Ray, S. S., Analytical Solution for the Space Fractional Diffusion Equation by Two-Step Adomian Decomposition Method, *Commun. Nonlinear Sci. Numer. Simul., 14* (2009), 4, pp. 1295-1306
[20] Meerschaert, M. M., *et al.*, Governing Equations and Solutions of Anomalous Random Walk Limits, *Phys. Rev. E, 66* (2002), 1, pp.102R-105R
[21] Metzler, R., Klafter, J., The Restaurant at the End of the Random Walk: Recent Developments in the Description of Anomalous Transport by Fractional Dynamics, *J. Phys. A, 37* (2004), 31, pp. R161-R208
[22] Hirota, R., Exact N-Solitons of the Wave Equation of Long Waves in Shallow-Water and in Nonlinear Lattices. *J. Math. Phy., 14* (1973), 7, pp. 810-814

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