

ON APPROXIMATE SOLUTIONS OF FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS

by

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The present paper is concerned with the implementation of optimal homotopy asymptotic method to handle the approximate analytical solutions of fractional partial differential equations. Approximate solutions of fractional models in both 1-D and 2-D cases are handled using the innovative proposed method. The consequences show excellent accuracy and strength of the planned method. Using this method, one can easily handle the convergence of approximation series solution for the fractional partial differential equations and can adjust the convergence region when required. The method is effective and explicit. Moreover, this method is flexible with respect to geometry and ease of implementation for fractional order models of physical and biological problems.

Key words: *optimal homotopy asymptotic method, analytical solutions, fractional partial differential equation*

Introduction

The application of fractional calculus is to describe numerous phenomenon in applied sciences, such as anomalous diffusion transport, fluid-flow in porous materials, dynamics in self-similar structures, acoustic wave propagation in viscoelastic materials, financial theory, signal processing, electric conductance of biological systems, and others, [1-7].

Many researchers have calculated the analytical solution of fractional differential equations using certain procedures like the Laplace transform Mellin transform, Fourier transform, method of variable separation, and other techniques [2, 8]. These are those exact analytical solutions of only a few easy cases and equivalent to a few functions such as the Fox H function and the hyperbolic geometric function [2, 8]. The reason of difficulties in finding exact solutions for most problems and the complexity of computing these special functions limits the applications of applied fractional differential equations in engineering and scientific computing fields. Most of the scholars are struggling and have developed numerical algorithms for solving the fractional PDE, including the finite difference method, finite element method, and spectral element method [9-16] to achieve the goal. We introduce the fractional PDE [16]:

$${}_0^C D_t^\alpha w(x, t) - \nabla^2 w(x, t) = g(x, t), \quad x \in \Omega, t \in [0, T] \quad (1)$$

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$$w(x, 0) = w_0, \quad x \in \Omega \quad (2)$$

where $\Omega \subset R^d$, $d = 1, 2, 3$ is a convex polygonal/polyhedral domain, and ∇^2 denotes the Laplacian. The ${}_0^C D_t^\alpha w(x, t)$ denotes the Caputo fractional order derivative and $g(x, t)$ is known function. Many physical phenomena can be modeled by eq. (1), for instance, the thermal diffusion in media with fractional geometry [10], highly heterogeneous aquifer [11], underground environmental problem [12], random walk [13, 14] ... etc.

- For $g(x, t) = [z_0(x, t)/\Gamma(1-\alpha)]t^{-\alpha}$ we retrieve the equivalent model of eq. (1) in form of fractional diffusion equation [13] with anomalous diffusion constant ($K_\alpha = 1$) that appears in many physical phenomena.
- For $\nabla^2 w(x, t) = \partial^2 w / \partial x^2$ and $g(x, t) = 0$ eq. (1) reduces into the form of generalized fractional transfer model [10] with $D_1 = 1$
- For $\nabla^2 w(x, t) = \partial^2 w / \partial x^2$ and $g(x, t) \neq 0$ eq. (1) reduces into the of fractional subdiffusion [15].

Theory of the optimal homotopy asymptotic method (OHAM)

We introduce OHAM [17] for fractional model, eq. (1):

$$\begin{aligned} D[w(x, t)] + g(x, t) &= 0, \quad x \in \delta, \quad t \geq 0 \\ \psi \left[w(x, t), \frac{\partial w(x, t)}{\partial t} \right] &= 0, \quad t \in \Omega \end{aligned} \quad (3)$$

where D denotes the operator which may be integer or fractional order differential operator, ψ denotes the boundary operator of eq. (3), $w(x, t)$ denotes the exact solution of eq. (3), x as well as t denote spatial and input free numeral, respectively, Ω denotes the collection of boundary points of δ , and $g(x, t)$ denotes known expression in eq. (3). Now, we can split the differential operator D into the term of L and N differential operators so that:

$$L[w(x, t)] + N[w(x, t)] + g(x, t) = 0, \quad x \in \delta$$

Here, L denotes the simpler linear differential operator which may be the linear and non-complicated portion of the eq. (3) so that it would be solvable via any auxiliary analytical method, whereas the operator N denotes the differential operator which would be non-linear and complicated portion of the eq. (3). Let us assume $w_0(x, t) : \delta \rightarrow R$ is the exact solution of:

$$\begin{aligned} L[w_0(x, t)] + g(x, t) &= 0 \\ \psi \left[\left(w_0(x, t), \frac{\partial w_0(x, t)}{\partial t} \right) \right] &= 0 \end{aligned} \quad (4)$$

and must be continuous function. The $w(x, t) : \delta \rightarrow R$ is the exact solution of eq. (3) that is also continuous. So, we can define a homotopy $F(x, t; q) : \delta \times [0, 1] \rightarrow R$ satisfies the equation:

$$(1 - q)\{L[F(x, t; q)] + g(x, t)\} = H(q)\{D[F(x, t; q)] + g(x, t)\} \quad (5)$$

In eq. (5), q belongs to the interval $[0, 1]$, that is embed auxiliary constant, $H(q)$ denotes an auxiliary expression of the eq. (3) so that $H(q)$ must be non-zero for all the points q except at $q = 0$. In addition, for $q = 0$ eq. (5) transfers into the formation of eq. (4) and for $q = 1$ eq. (5) transfers into the formation of eq. (3). According to the definition of homotopy, we get:

$$F(x, t; q) = w_0(x, t), \text{ at } q = 0, \quad F(x, t; q) = w(x, t), \text{ at } q = 1$$

Here, we choose the auxiliary expression $H(q)$ for substituting into eq. (5) so that:

$$H(q) = C_1q + C_2q^2 + C_3q^3 + \dots + C_kq^k + \dots,$$

where

$$C_1, C_2, C_3, \dots, C_k, \dots$$

are auxiliary constants.

According to Taylor's expansion we can expand $F(x, t; q, C_1, C_2, \dots)$ so that:

$$F(x, t; q, C_1, C_2, \dots) = w_0(x, t) + \sum_{r=1}^{\infty} w_r(x, t; C_1, C_2, \dots, C_r) q^r \quad (6)$$

We substitute $H(q)$ and $F(x, t; q, C_1, C_2, \dots)$ in eq. (5) and equate coefficient of the same powers of q after expanding eq. (5). we obtain zeroth-order eq. (4) and series problems, respectively:

$$L[w_1(x, t)] = C_1 N_0[w_0(x, t)], \quad \psi \left[w_1(x, t), \frac{\partial w_1(x, t)}{\partial t} \right] = 0$$

and

$$L[w_2(x, t)] = C_2 N_0[w_0(x, t)] + C_1 N_1[w_0(x, t), w_1(x, t)] + (1 + C_1) L[w_1(x, t)] \\ \psi \left[w_2(x, t), \frac{\partial w_2(x, t)}{\partial t} \right] = 0$$

Moreover, the general governing r^{th} -order problem of the analytical solution $w_r(x, t)$ is in the form:

$$L[w_r(x, t)] = L[w_{r-1}(x, t)] + C_r N_0[w_0(x, t)] + \sum_{j=1}^{r-1} C_j \left\{ L[w_{r-j}(x, t)] \right. \\ \left. + N_{r-j}[w_0(x, t), w_1(x, t), \dots, w_{r-j}(x, t)] \right\} \\ \psi \left[w_r(x, t), \frac{\partial w_r(x, t)}{\partial t} \right] = 0, \quad r = 2, 3, \dots$$

Consequently, the series (6) converges at $q = 1$:

$$w(x, t; C_1, C_2, \dots) = z_0(x, t) + \sum_{i \geq 1} w_i(x, t; C_1, C_2, \dots, C_i) \quad (7)$$

The residual:

$$R(x, t; C_1, C_2, \dots) = D[w(x, t; C_1, C_2, \dots)] + g(x, t) \quad (8)$$

Least square method for to compute the value of auxiliary constants C_1, C_2, C_3, \dots , as:

$$j(C_1, C_2, C_3, \dots) = \int_0^t \int_{\delta} R^2(x, t; C_1, C_2, \dots) dx dt$$

and then use the system of equations:

$$\frac{\partial j}{\partial C_1} = \frac{\partial j}{\partial C_2} = \frac{\partial j}{\partial C_3} = \dots = 0$$

Numerical simulations

In this section, we apply OHAM for four problem that are 1-D and 2-D:

Example 1. Let us consider the fractional PDE [15, 16] of 1-D:

$${}_0^c D_t^\alpha w(x,t) - \frac{\partial^2 w(x,t)}{\partial x^2} = f(x,t), \quad t \in [0, T], \quad x \in \Omega, \quad \alpha = \frac{1}{2} \quad (9)$$

with the exact solution:

$$w(x,t) = t^m [\sin(2\pi x)], \quad m = 3.5$$

and initial condition:

$$w(x,0) = 0$$

We are opting differential operators for eq. (9):

$$L[F(x,t;q)] = {}_0^c D_t^\alpha F(x,t;q), \quad N[F(x,t;q)] = -\frac{\partial^2 F(x,t;q)}{\partial x^2}$$

with the initial condition:

$$F(x,0;q) = 0$$

Following the fundamental concept of OHAM, we begin with:

– zeroth-order problem

$$\frac{\partial^{1/2} w_0(x,t)}{\partial t^{1/2}} = 0, \quad w_0(x,0) = 0$$

Its solution is:

$$w_0(x,t) = [19.1334263t^4 + 1.0t^{7/2}] \sin(z) \quad (10)$$

wher $z = 6.2831853x$

– first-order problem

$$\frac{\partial^{1/2} w_1(x,t)}{\partial t^{1/2}} = C_1 N_0 [w_0(x,t)], \quad w_1(x,0) = 0$$

Its solution is:

$$w_1(x,t) = [9.1334263t^4 + 346.3434348t^{9/2}] C_1 \sin(z) \quad (11)$$

– second-order problem

$$\frac{\partial^{1/2} w_2(x,t)}{\partial t^{1/2}} = C_2 N_0 [w_0(x,t)] + C_1 N_1 [w_0(x,t), w_1(x,t)] + (1 + C_1) L_1 [w_1(x,t)], \quad w_2(x,0) = 0$$

It solution is:

$$w_2(x, t) = \sin(z) \left[\begin{array}{l} 346.343t^{9/2}(C_1^2 + C_2) + C_1(C_1 + 1)(19.133t^4 + 346.343t^{9/2}) \\ + 19.133C_2t^4 + 5964.063C_1^2t^5 \end{array} \right] \quad (12)$$

– third-order problem:

$$\frac{\partial^{1/2} w_3(x, t)}{\partial t^{1/2}} = C_3 N_0 [w_0(x, t)] + C_2 \{ N_2 [w_0(x, t), w_1(x, t), w_2(x, t)] + L_1 [w_1(x, t)] \} \\ + C_1 N_0 (w_0, w_1) + (1 + C_1) L_2 [w_2(x, t)], \quad w_3(x, 0) = 0$$

Its solution is:

$$w_3(x, t) = \sin(z) \left\{ \begin{array}{l} (C_1 + 1)[346.343(C_1^2 + C_2)t^{9/2} + C_1(C_1 + 1)(19.133t^4 + 346.343t^{9/2}) \\ + 38.267C_2t^4 + 5964.063C_1^2t^5] + 346.343t^{9/2}[C_3 + C_1C_2 + C_1^2(C_1 + 1) + C_1C_2] \\ + C_1[5964.1t^5(C_1^2 + C_2) + 5964.063t^5C_1(C_1 + 1)] + 19.133C_3t^4 \\ + 98143.998C_1^3t^{11/2} + 5964.0629C_1C_2t^5 \end{array} \right\} \quad (13)$$

Adding eq. (10) to eq. (13), we obtain:

$$w(x, t) = \sin(z) \left\{ \begin{array}{l} (C_1 + 1)[346.343t^{9/2}(C_1^2 + C_2) + C_1(C_1 + 1)(19.133t^4 + 346.343t^{9/2}) \\ + 19.133C_2t^4 + 5964.0629C_1^2t^5] + 346.343t^{9/2}[C_3 + 2C_1C_2 + C_1^2(C_1 + 1)] \\ + 346.343t^{9/2}(C_1^2 + C_2) + 19.133t^4 + t^{7/2} + 0.2t^5C_1(29820.315\sin(z)C_1^2 + 1) \\ \cdot (19.133C_1t^4 + 346.343C_1t^{9/2}) + 19.133C_1t^4 + 19.133C_2t^4 + 19.133C_3t^4 \\ + 346.343C_1t^{9/2} + 5964.063C_1^2t^5 + 98143.998C_1^3t^{11/2} + 5964.063C_1C_2t^5 \end{array} \right\} \quad (14)$$

values of auxiliary constants are obtained by using collocation method:

$$C_1 = -0.041191063954626 \\ C_2 = 0.0012441310732261 \\ C_3 = -0.000011636101092243$$

We substitute the auxiliary constants values into eq. (14), we obtained third order approximation

$$w(x, t) = 16.9t^4 \sin(6.28x) + 28.5t^5 \sin(6.28x) + t^{7/2} \sin(6.28x) \\ - 38.5t^{9/2} \sin(6.28x) - 6.86t^{11/2} \sin(6.28x)$$

Table 1 shows the OHAM approximation, exact solution and absolute errors. Whereas, the absolute error has been measured by $L_\infty = |w_{\text{exact}}(x, t) - w_{\text{OHAM}}(x, t)|$. The OHAM solution, exact solution, and absolute errors are plotted for various values of x and t in figs. 1 and 2 and curves of both OHAM and exact solution are exactly matched.

Table 1. Various numerical results of example 1 using OHAM

x	t = 1			t = 2			t = 3		
	Exact	OHAM	L_∞	Exact	OHAM	L_∞	Exact	OHAM	L_∞
0.00	0.0000	0.0000	0.0000	0.00000	0.00000	0.0000	0.00000	0.00000	0.0000
0.10	0.5878	0.5921	0.0043	6.65000	6.62700	0.0230	27.4900	27.6400	0.1480
0.20	0.9511	0.9580	0.0069	10.7600	10.7200	0.0371	44.4800	44.7200	0.2400
0.30	0.9511	0.9580	0.0069	10.7600	10.7200	0.0371	44.4800	44.7200	0.2400
0.40	0.5878	0.5921	0.0043	6.65000	6.62700	0.0230	27.4900	27.6400	0.1480
0.50	0.0000	0.0000	0.0000	0.00000	0.00000	0.0000	0.00000	0.0000	0.0000
0.60	0.5878	0.5921	0.0043	6.65000	6.62700	0.0230	27.4900	27.6400	0.1480
0.70	0.9511	0.9580	0.0069	10.7600	10.7200	0.0371	44.4800	44.7200	0.2400
0.80	0.9511	0.9580	0.0069	10.7600	10.7200	0.0371	44.4800	44.7200	0.2400
0.90	0.5878	0.5921	0.0043	6.65000	6.62700	0.0230	27.4900	27.6400	0.1480
1.00	0.0000	0.0000	0.0000	0.00000	0.00000	0.0000	0.00000	0.0000	0.0000

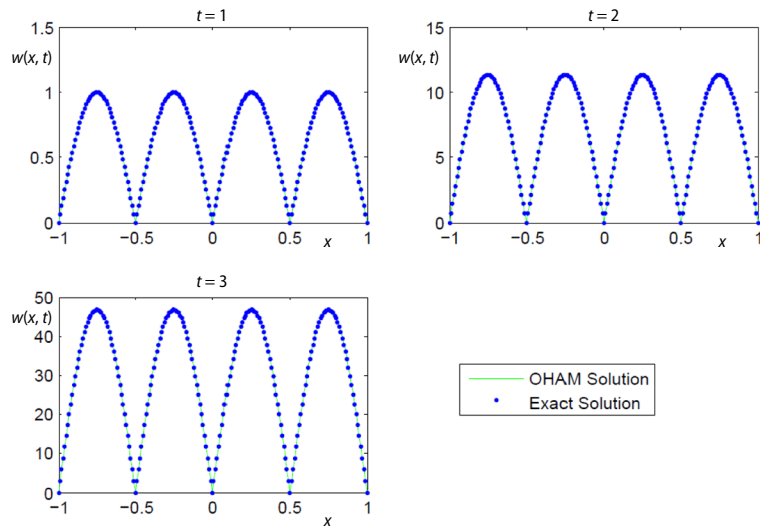


Figure 1. The OHAM solution and exact solution example 1 for different values of x and t (for color image see journal web site)

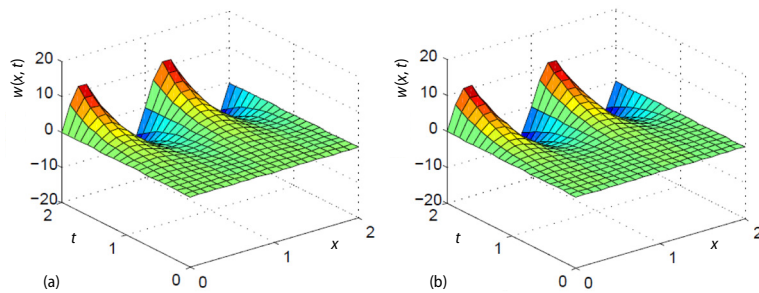


Figure 2. The OHAM solution and exact solution example 1 for different values of x and t; (a) OHAM solution and (b) exact solution (for color image see journal web site)

Example 2. Consider the fractional PDE [15,16] of 1-D:

$${}_0^C D_t^\alpha w(x,t) - \frac{\partial^2 w(x,t)}{\partial x^2} = f(x,t), \quad t \in [0, T], \quad x \in \Omega, \quad (15)$$

$$w(x,0) = 0, \quad x \in \Omega, \quad w(0,t) = w(1,t) = 0, \quad t \in [0, T],$$

where

$$f(x,t) = e^x \frac{(5+\alpha)}{24} t^4 - t^{4+\alpha} e^x, \quad \alpha = \frac{1}{2}$$

We are opting the differential operators for eq. (15):

$$L[F(x,t;q)] = {}_0^C D_t^\alpha F(x,t;q), \quad N[F(x,t;q)] = -\frac{\partial^2 F(x,t;q)}{\partial x^2}$$

We follow the same working rules of OHAM as applied for *example 1*, we got:

$$w_0(x,t) = [t^{9/2} - 0.4362t^5] \exp(x), \quad w_1(x,t) = [0.18181818t^{11/2} - 0.43618981t^5] C_1 \exp(x)$$

$$w_2(x,t) = \exp(x) \left\{ \begin{array}{l} 0.417t^{11/2} (0.436C_1^2 + 0.436C_2) - (C_1 + 1)(0.436C_1t^5 - 0.182C_1t^{11/2}) \\ -0.436C_2t^5 - 0.0723C_1^2t^6 \end{array} \right\}$$

In same manner, we can find the solution $w_3(x,t)$. Now we obtain:

$$w(x,t) = w_0(x,t) + w_1(x,t) + w_2(x,t) + w_3(x,t) \quad (16)$$

We apply the collocation method and computed:

$$C_1 = -1.4531461332285 \quad C_2 = -0.014911625753517 \quad C_3 = 0.030940407558591$$

Substituting the auxiliary constants values into eq. (16), we obtained the approximation of OHAM:

$$w(x,t) = 0.03065t^5 e^x + 0.2055t^6 e^x + 1.0t^{9/2} e^x - 0.1507t^{11/2} e^x - 0.08583t^{13/2} e^x$$

Table 2 shows the OHAM approximation, exact solution and absolute errors. The OHAM solution, exact solution, and absolute errors are plotted for different values of x and t in figs. 3 and 4 and curves of both OHAM and exact solution are exactly matched.

Example 3. Consider the fractional PDE [16] of 2-D:

$${}_0^C D_t^\alpha w(x,t) - \frac{\partial^2 w(x,t)}{\partial x_1^2} - \frac{\partial^2 w(x,t)}{\partial x_2^2} = f(x,t), \quad t \in [0, T], \quad x \in \Omega, \quad \text{where } \alpha = \frac{1}{2} \quad (17)$$

With the exact solution:

$$w(x,t) = t^m [\sin(2\pi x_1)][\sin(2\pi x_2)], \quad m = 3.5$$

and initial condition:

$$w(x,0) = 0$$

Let say $z_1 = 6.2831853x_1$ and $z_2 = 6.2831853x_2$. Now, we opt the operators for eq. (17)

$$L[F(x,t;q)] = {}_0^C D_t^\alpha F(x,t;q), \quad N[F(x,t;q)] = -\frac{\partial^2 F(x,t;q)}{\partial x^2}$$

Table 2. Various numerical results of example 2 using OHAM

x	t = 0.1			t = 0.2			t = 1		
	Exact	OHAM	$L_{\infty} \cdot 10^{-8}$	Exact	OHAM	$L_{\infty} \cdot 10^{-6}$	Exact	OHAM	L_{∞}
0.0	0.0000316	0.0000316	0.821	0.000715542	0.000714475	1.070	1.0000	0.9996	0.0004
0.2	0.0000386	0.0000386	1.000	0.000873965	0.000872662	1.300	1.2210	1.2210	0.0005
0.4	0.0000472	0.0000472	1.220	0.001067460	0.001065870	1.590	1.4920	1.4910	0.0006
0.6	0.0000576	0.0000576	1.500	0.001303800	0.001301860	1.940	1.8220	1.8210	0.0007
0.8	0.0000704	0.0000704	1.830	0.001592470	0.001590090	2.370	2.2260	2.2250	0.0008
1.0	0.0000860	0.0000860	2.230	0.001945040	0.001942150	2.900	2.7180	2.7170	0.0010
1.2	0.0001050	0.0001050	2.730	0.002375680	0.002372140	3.540	3.3200	3.3190	0.0013
1.4	0.0001282	0.0001283	3.330	0.002901660	0.002897340	4.320	4.0550	4.0540	0.0015
1.6	0.0001566	0.0001567	4.070	0.003544100	0.003538820	5.280	4.9530	4.9510	0.0019
1.8	0.0001913	0.0001914	4.970	0.004328780	0.004322320	6.450	6.0500	6.0470	0.0023
2.0	0.0002337	0.0002337	6.070	0.005287180	0.005279300	7.880	7.3890	7.3860	0.0028

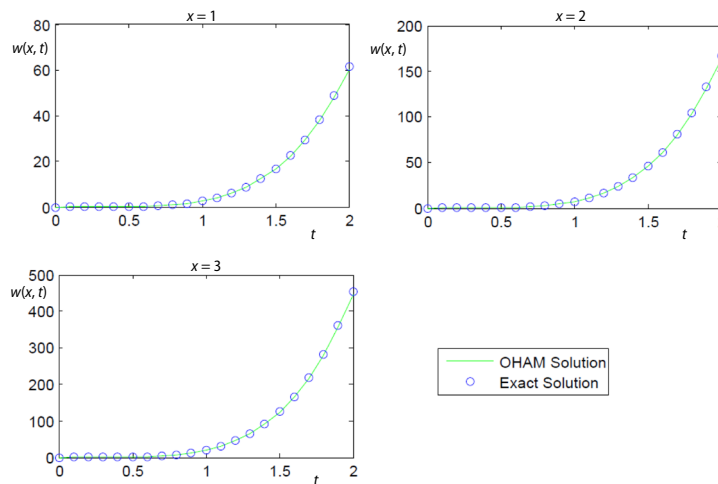


Figure 3. The OHAM solution and exact solution example 2 for different values of x and t (for color image see journal web site)

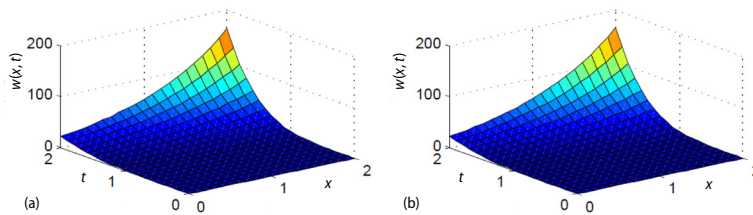


Figure 4. The OHAM solution and exact solution example 2 for different values of x and t; (a) OHAM solution and (b) exact solution (for color image see journal web site)

Following the same working rules of OHAM as applied for example 1, we got:

$$w_0(x,t) = [38.267t^4 + 1.0t^{7/2}] \sin(z_1) \sin(z_2)$$

$$w_1(x, t) = [38.267t^4 + 1385.374t^{9/2}]C_1 \sin(z_1) \sin(z_2)$$

$$w_2(x, t) = \sin(z_1) \sin(z_2) \left\{ \begin{aligned} &(C_1 + 1.0)(38.266853C_1 t^4 + 1385.3737C_1 t^{9/2}) \\ &+ 0.45851598 t^{9/2} (3021.4296 C_1^2 + 3021.4296 C_2) \\ &+ 47712.503 C_1^2 t^5 + 38.267 C_2 t^4 \end{aligned} \right\}$$

In same manner, we can find the solution $w_3(x, t)$. Hence, we obtain:

$$w(x, t) = w_0(x, t) + w_1(x, t) + w_2(x, t) + w_3(x, t) \tag{18}$$

we computed:

$$C_1 = -0.04123982834584 \quad C_2 = -0.004192871561469 \quad C_3 = -0.0033155516059915$$

and by substitution the auxiliary constants values into eq. (18), third order approximate solution can be obtained:

$$w(x, t) = 33.28t^4 \sin(6.283x_1) \sin(6.283x_2) + 249.9t^5 \sin(6.283x_1) \sin(6.283x_2) \\ + 1.0t^{7/2} \sin(6.283x_1) \sin(6.283x_2) - 173.0t^{9/2} \sin(6.283x_1) \sin(6.283x_2) \\ - 110.1t^{11/2} \sin(6.283x_1) \sin(6.283x_2)$$

Table 3 and figs. 5 and 6 show the corresponding results of OHAM approximate solution of the *example 3*.

Table 3. Various numerical results of *example 3* using OHAM

x_1	$t = 0.1, x_2 = 0.1$			$t = 1, x_2 = 1.1$			$t = 1, x_2 = 1.2$		
	Exact $\cdot 10^{-3}$	OHAM $\cdot 10^{-3}$	$L_\infty \cdot 10^{-6}$	Exact	OHAM	$L_\infty \cdot 10^{-4}$	Exact	OHAM	$L_\infty \cdot 10^{-4}$
-10.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001
-9.19	0.1728	0.1766	3.790	0.5465	0.5468	2.640	0.8843	0.8847	4.270
-8.38	0.1272	0.1300	2.790	0.4024	0.4026	1.940	0.6510	0.6514	3.140
-7.57	0.0791	0.0809	1.740	0.2503	0.2504	1.210	0.4049	0.4051	1.950
-6.76	0.1855	0.1896	4.070	0.5866	0.5869	2.830	0.9492	0.9496	4.580
-5.95	0.0574	0.0587	1.260	0.1816	0.1817	0.876	0.2939	0.2940	1.420
-5.14	0.1432	0.1464	3.140	0.4529	0.4531	2.180	0.7328	0.7332	3.530
-4.33	0.1629	0.1665	3.580	0.5151	0.5153	2.480	0.8334	0.8338	4.020
-3.52	0.0233	0.0238	0.512	0.0737	0.0737	0.355	0.1192	0.1193	0.575
-2.71	0.1800	0.1840	3.950	0.5693	0.5696	2.750	0.9212	0.9216	4.440
-1.90	0.1093	0.1117	2.400	0.3455	0.3457	1.670	0.5590	0.5593	2.700
-1.09	0.0996	0.1018	2.190	0.3150	0.3151	1.520	0.5096	0.5098	2.460
-0.28	0.1826	0.1866	4.010	0.5774	0.5777	2.780	0.9342	0.9347	4.510
0.53	0.0348	0.0356	7.65	0.1101	0.1102	5.310	0.1782	0.1783	0.860
1.34	0.1569	0.1604	3.450	0.4963	0.4965	2.390	0.8030	0.8034	3.870
2.15	0.1504	0.1537	3.300	0.4755	0.4758	2.290	0.7694	0.7698	3.710
2.96	0.0462	0.0472	1.010	0.1462	0.1462	0.705	0.2365	0.2366	1.140
3.77	0.1844	0.1885	4.050	0.5832	0.5834	2.810	0.9436	0.9440	4.550
4.58	0.0895	0.0915	1.970	0.2832	0.2833	1.370	0.4582	0.4584	2.210

Figure 5. The OHAM solution and exact solution *example 3* for different values of $x = (x_1, x_2)$ and t (for color image see journal web site)

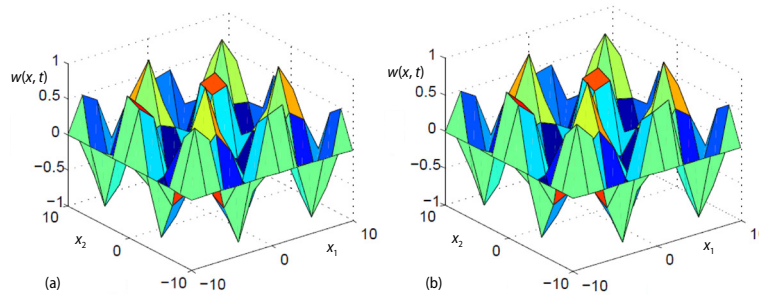
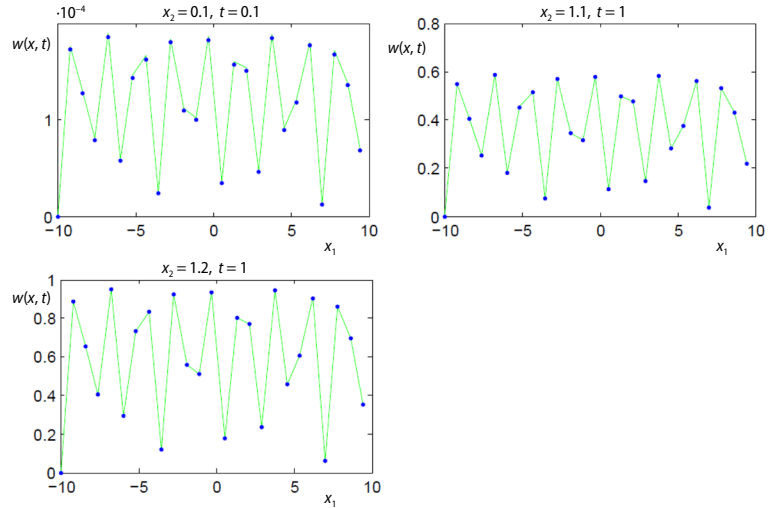


Figure 6. The OHAM solution and exact solution *example 3* at $t = 1$ and for different values of $x = (x_1, x_2)$; (a) OHAM solution and (b) exact solution (for color image see journal web site)

Example 4. Consider the fractional PDE [16] of 2-D:

$${}_0^c D_t^\alpha w(x,t) - \frac{\partial^2 w(x,t)}{\partial x_1^2} - \frac{\partial^2 w(x,t)}{\partial x_2^2} = f(x,t), \quad t \in [0, T], \quad x \in \Omega, \quad \alpha = \frac{1}{2} \quad (19)$$

We follow the same working rules of OHAM as applied for *example 1*, we got:

$$w_0(x, t) = [t^{9/2} - 0.4362t^5] \exp(x_1 + x_2)$$

$$w_1(x, t) = C_1 [0.36364t^{11/2} - 0.8724t^5] \exp(x_1 + x_2)$$

$$w_2(x, t) = \left\{ \begin{array}{l} 0.41683271t^{11/2} (1.7447593C_1^2 + 0.8723793C_2) \\ -(C_1 + 1)(0.87237963C_1t^5 - 0.36363636C_1t^{11/2}) \\ -0.87237963C_2t^5 - 0.29079321C_1^2t^6 \end{array} \right\} \exp(x_1 + x_2)$$

In same manner, we can find the solutions $w_3(x, t)$ and $w_4(x, t)$. Therefore, we obtain:

$$w(x, t) = w_0(x, t) + w_1(x, t) + w_2(x, t) + w_3(x, t) + w_4(x, t). \tag{20}$$

We apply collocation method and computed:

$$C_1 = 0.19889044496277 \quad C_2 = -1.1677179805875$$

$$C_3 = 1.2149520660442 \quad C_4 = -0.22776877939531$$

We obtained the approximate OHAM solution:

$$\tilde{w}(x, t) = 0.21t^5 e^{x_1+x_2} (x_1 + x_2) - 0.36t^6 e^{x_1+x_2} + 1.5e - 3t^7 e^{x_1+x_2}$$

$$+ 1.0t^{9/2} e^{x_1+x_2} + 0.049t^{11/2} e^{x_1+x_2} + 0.086t^{13/2} e^{x_1+x_2}$$

Table 4 and figs. 7 and 8 show the corresponding results of OHAM approximate solution of the *example 4*.

Table 4. Various Numerical results of *example 4* using OHAM

x_1	$t = 0.01, x_2 = 0.01$			$t = 1, x_2 = 0.01$			$t = 0.01, x_2 = 0.1$		
	Exact	OHAM	L_∞	Exact $\cdot 10^{-8}$	OHAM $\cdot 10^{-8}$	$L_\infty \cdot 10^{-11}$	Exact $\cdot 10^{-8}$	OHAM $\cdot 10^{-8}$	$L_\infty \cdot 10^{-11}$
-1.0	0.3716	0.3649	0.0067	0.0372	0.0379	0.770	0.0407	0.0415	0.840
-0.9	0.4107	0.4032	0.0074	0.0411	0.0419	0.850	0.0449	0.0459	0.930
-0.8	0.4538	0.4456	0.0082	0.0454	0.0463	0.940	0.0497	0.0507	1.000
-0.7	0.5016	0.4925	0.0091	0.0502	0.0512	1.000	0.0549	0.0560	1.100
-0.6	0.5543	0.5443	0.0100	0.0554	0.0566	1.200	0.0607	0.0619	1.300
-0.5	0.6126	0.6016	0.0110	0.0613	0.0625	1.300	0.0670	0.0684	1.400
-0.4	0.6771	0.6648	0.0120	0.0677	0.0691	1.400	0.0741	0.0756	1.500
-0.3	0.7483	0.7347	0.0140	0.0748	0.0764	1.600	0.0819	0.0836	1.700
-0.2	0.8270	0.8120	0.0150	0.0827	0.0844	1.700	0.0905	0.0924	1.900
-0.1	0.9139	0.8974	0.0170	0.0914	0.0933	1.900	0.1000	0.1021	2.100
0.0	1.0100	0.9918	0.0180	0.1010	0.1031	2.100	0.1105	0.1128	2.300
0.1	1.1160	1.0960	0.0200	0.1116	0.1139	2.300	0.1221	0.1247	2.500
0.2	1.2340	1.2110	0.0220	0.1234	0.1259	2.600	0.1350	0.1378	2.800
0.3	1.3630	1.3390	0.0250	0.1363	0.1392	2.800	0.1492	0.1523	3.100
0.4	1.5070	1.4800	0.0270	0.1507	0.1538	3.100	0.1649	0.1683	3.400
0.5	1.6650	1.6350	0.0300	0.1665	0.1700	3.500	0.1822	0.1860	3.800
0.6	1.8400	1.8070	0.0330	0.1840	0.1879	3.800	0.2014	0.2056	4.200
0.7	2.0340	1.9970	0.0370	0.2034	0.2076	4.200	0.2226	0.2272	4.600
0.8	2.2480	2.2070	0.0410	0.2248	0.2295	4.700	0.2460	0.2511	5.100
0.9	2.4840	2.4390	0.0450	0.2484	0.2536	5.200	0.2718	0.2775	5.600
1.0	2.7460	2.6960	0.0500	0.2746	0.2803	5.700	0.3004	0.3067	6.200

Conclusion

In this work, a recently developed analytical technique known as OHAM has been employed to find the approximate solution of fractional PDE. This analytical technique fully satisfies the physical behavior of the fractional models and provides us a convenient approach to control the convergence of approximate solution to exact solution. Excellent accuracy of the approximate solution is shown. It is evidenced from simulation that the approximate solution of fractional models has agreement with the exact solution. This is a significant progress in computing the solution of time fractional PDE.

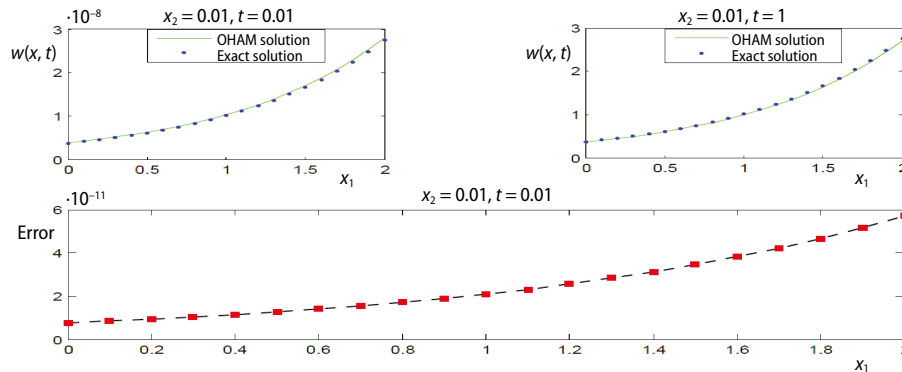


Figure 7. The OHAM solution, exact solution and error of *example 4* for different values $x = (x_1, x_2)$ and t

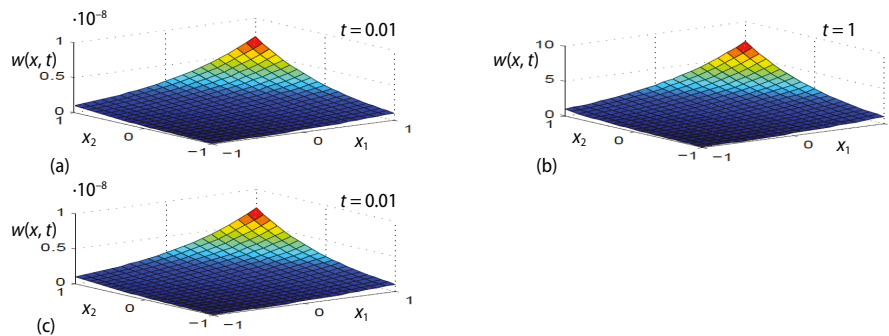


Figure 8. The OHAM solution and exact solution of *example 4* for different values of $x = (x_1, x_2)$ and t ; (a) OHAM solution for $t = 0.01$, (b) OHAM solution for $t = 1$, and exact solution for $t = 0.01$ (for color image see journal web site)

References

- [1] Oldham, K. B., Spanier, J., *The Fractional PD* York, USA, 1974
- [2] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, Cal., USA, 1999
- [3] Kilbas, A., et al., *Theory and Applications of Fractional Differential Equations*, Elsevier, Boston, Mass., USA, 2006
- [4] Mainardi, F., *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, 2010
- [5] Anastassiou, G.-A., *Advances on Fractional Inequalities*, Springer, New York, USA, 2011
- [6] Odibat, Z., Computing Eigenlements of Boundary Value Problems with Fractional Derivatives, *Appl. Math. Comput.*, 215 (2009), 8, pp. 3017-3028
- [7] M. Stojanović, Numerical Method for Solving Diffusion-Wave Phenomena, *J. Comput. Appl. Math.*, 235 (2011), 10, pp. 3121-3137
- [8] Mainardi, F., et al., Fox H Functions in Fractional Diffusion, *J. Comput. Appl. Math.*, 178 (2005), 1-2, pp. 321-331
- [9] Yuste, S.-B., Weighted Average Finite Difference Methods for Fractional Diffusion Equations, *J. Comput. Phys.*, 216 (2006), 1, pp. 264-274
- [10] Nigmatulin, R., The Realization of the Generalized Transfer Equation in a Medium with Fractal Geometry, *Phys. Stat. Sol. B*, 133 (1986), 1, pp. 425-430
- [11] Adams, E. E., Lynn, W. G., Field Study of Dispersion in a Heterogeneous Aquifer: 2. Spatial Moments Analysis, *Water Resources Research*, 28 (1992), 12, pp. 3293-3307

- [12] Hatano, Y., Hatano, N., Dispersive Transport of Ions in Column Experiments: An Explanation of Long-Tailed profiles, *Water Resources Research*, 34 (1998), 5, pp. 1027-1033
- [13] Metzler, R., Klafter, J., The Restaurant at the end of the Random Walk: Recent Developments in the Description of Anomalous Transport by Fractional Dynamics, *J. Phys. A Math. Gen.*, 37 (2004), 31, pp. 161-208
- [14] Gorenflo, R., Mainardi, F., Random Walk Models for Space Fractional Diffusion Processes. *Fract. Calc. Appl. Anal.*, 1 (1998), 2, pp. 167-191
- [15] Gao, G. H., *et al.*, New Fractional Numerical Differentiation Formula to Approximate the Caputo Fractional Derivative and its Applications. *J. Comput. Phys.*, 259 (2014), Feb., pp. 33-50
- [16] Li, Z., *et al.*, High-Order Numerical Methods for Solving Time Fractional Partial Differential Equations, *Journal of Scientific Computing*, 2016 (2016), 2, pp. 1-19
- [17] Marinca, V., Herisanu, N., Application of Optimal Homotopy Asymptotic Method for Solving Nonlinear Equations Arising in Heat Transfer, *Int. Commun. Heat Mass Transf.*, 35 (2008), 6, pp. 710-715