A NOVEL METHOD FOR THE SPACE AND TIME FRACTIONAL BLOCH-TORREY EQUATIONS

by

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Reproducing kernel technique was implemented to solve the fractional Bloch-Torrey equations. This efficient technique was used via some useful reproducing kernel functions, to obtain approximations to the exact solution in form of series solutions. A numerical example has been presented to prove efficiency of developed technique.

Key words: reproducing kernel technique, fractional Bloch-Torrey equations

Introduction

We consider the fractional Bloch-Torrey equation in the reproducing kernel Hilbert space:

$$ {}_0^C\mathrm{D}_\tau^\alpha v(y,\tau) = K\frac{\partial^\beta v(y,\tau)}{\partial|y|^\beta} + f(y,\tau), \quad y\in(a,b), \quad \tau\in(0,T] $$

$$ v(y,0)=\omega(y), \quad y\in(a,b), \quad v(a,\tau)=0, \quad v(b,\tau)=0, \quad \tau\in(0,T] $$

where $0<\alpha\le 1$, $1<\beta\le 2$, $\omega(a)=0, \omega(b)=0$ and K is the generalized diffusion coefficient, ${}_0^C\mathrm{D}_\tau^\alpha v(y,\tau)$ defines Caputo fractional derivative of order α and is given:

$$ {}_0^C\mathrm{D}_\tau^\alpha v(y,\tau) = \frac{1}{\Gamma(1-\alpha)}\int_0^\tau \frac{\partial u(y,s)}{\partial s}\frac{1}{(\tau-s)^\alpha}\mathrm{d}s $$

and $\partial^\beta v(y,\tau)/\partial|y|^\beta$ is Riesz fractional derivative of order β and presented:

$$ \frac{\partial^\beta v(y,\tau)}{\partial|y|^\beta} = -\frac{1}{2\cos\left(\frac{\pi\beta}{2}\right)}({}_a\mathrm{D}_y^\beta + {}_y\mathrm{D}_b^\beta)v(y,\tau) $$

with

$$ {}_a\mathrm{D}_y^\beta v(y,\tau) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial y^2}\int_a^y \frac{v(\xi,\tau)}{(y-\xi)^{\beta-1}}\mathrm{d}\xi $$

and

$$ {}_y\mathrm{D}_b^\beta v(y,\tau) = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial y^2}\int_y^b \frac{v(\xi,\tau)}{(\xi-y)^{\beta-1}}\mathrm{d}\xi $$

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the left Riemann-Liouville fractional derivative and right Riemann-Liouville fractional derivative, respectively [1].

Reproducing kernel method was used to investigate the 1-D space and time fractional Bloch-Torrey equations. The concept of reproducing kernel has been given by Zaremba [2]. Mercer has presented the following inequality [3]:

$$\sum_{p,q=1}^{n} m(x_p, t_q)\xi_i\xi_j \geq 0$$

He introduced the reproducibility of the kernel as:

$$h(s) = \langle h(y), m(y,s) \rangle$$

Aronszajn [4] reduced the studies and showed a systematic reproducing kernel theory containing the Bergman kernel function. For more details see [5-10].

Reproducing kernel functions and main results

Definition 1. $W_2^1[0,1]$ is given as:

$$W_2^1[0,1] = \{f \in AC[0,1] : f' \in L^2[0,1]\}$$

$$\langle f, g \rangle_{W_2^1} = \int_0^1 [f(\eta)g(\eta) + f'(\eta)g'(\eta)]\mathrm{d}\eta, \quad f, g \in W_2^1[0,1]$$

and

$$\|f\|_{W_2^1} = \sqrt{\langle f, f \rangle_{W_2^1}}, \quad f \in W_2^1[0,1]$$

are the inner product and the norm in $W_2^1[0,1]$. The kernel function $T_\eta(\varsigma)$ of $W_2^1[0,1]$ is given [3]:

$$T_\eta(\varsigma) = \frac{1}{2\sinh(1)}\left[\cosh(\eta + \varsigma - 1) + \cosh(|\eta - \varsigma| - 1)\right]$$

Definition 2. We define the space $W_2^3[0,1]$:

$$W_2^3[0,1] = \{p \in AC[0,1] : p', p'' \in AC[0,1], \quad p^{(3)} \in L^2[0,1], p(0) = p(1) = 0\}$$

The inner product and the norm in $W_2^3[0,1]$ are given:

$$\langle p, q \rangle_{W_2^3} = \sum_{i=0}^{2} p^{(i)}(0)q^{(i)}(0) + \int_0^1 p^{(3)}(y)q^{(3)}(y)\mathrm{d}y, \quad p, q \in W_2^3[0,1]$$

and

$$\|p\|_{W_2^3} = \sqrt{\langle p, p \rangle_{W_2^3}}, \quad p \in W_2^3[0,1]$$

Theorem 3. The kernel function R_z of $W_2^3[0,1]$ is given:

$$R_z(y) = \begin{cases} \sum_{i=1}^{5} c_i(z)y^i, & y \leq z \\ \sum_{i=0}^{5} d_i(z)y^i, & y > z \end{cases}$$

where

$$c_1(z)=\frac{5}{156}z^4-\frac{1}{156}z^5-\frac{5}{26}z^2-\frac{5}{78}z^3+\frac{3}{13}z$$

$$c_2(z)=\frac{5}{624}z^4-\frac{1}{624}z^5+\frac{21}{104}z^2-\frac{5}{312}z^3-\frac{5}{26}z$$

$$c_3(z)=\frac{5}{1872}z^4-\frac{1}{1872}z^5+\frac{7}{104}z^2-\frac{5}{936}z^3-\frac{5}{78}z$$

$$c_4(z)=-\frac{5}{3744}z^4+\frac{1}{3744}z^5+\frac{5}{624}z^2+\frac{5}{1872}z^3-\frac{1}{104}z$$

$$c_5(z)=\frac{5}{3744}z^4-\frac{1}{18720}z^5-\frac{1}{624}z^2-\frac{1}{1872}z^3-\frac{1}{156}z+\frac{1}{120}$$

$$d_0(z)=\frac{1}{120}z^5,\quad d_1(z)=-\frac{1}{104}z^4-\frac{1}{156}z^5-\frac{5}{26}z^2-\frac{5}{78}z^3+\frac{3}{13}z$$

$$d_2(z)=\frac{7}{104}z^3+\frac{5}{624}z^4-\frac{1}{624}z^5+\frac{21}{104}z^2-\frac{5}{26}z$$

$$d_3(z)=\frac{5}{1872}z^4-\frac{1}{1872}z^5-\frac{5}{312}z^2-\frac{5}{936}z^3-\frac{5}{78}z$$

$$d_4(z)=-\frac{5}{3744}z^4+\frac{1}{3744}z^5+\frac{5}{624}z^2+\frac{5}{1872}z^3+\frac{5}{156}z$$

$$d_5(z)=\frac{1}{3744}z^4-\frac{1}{18720}z^5-\frac{1}{624}z^2-\frac{1}{1872}z^3-\frac{1}{156}z$$

Definition 4. We present the binary reproducing kernel Hilbert space $W(\Omega)$:

$$W(\Omega)=\left\{\begin{array}{l} v:\dfrac{\partial^3 v}{\partial y^2\partial t}\ \text{ is completely continuous in }\ \Omega=[0,1]\times[0,1], \\ \dfrac{\partial^5 v}{\partial y^3\partial t^2}\in L^2(\Omega),\quad v(y,0)=0,\quad v(0,t)=0, v(1,t)=0 \end{array}\right\}$$

The inner product and the norm in $W(\Omega)$ are given:

$$\langle v,g\rangle_W=\sum_{i=0}^{2}\int_0^1\left[\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial y^i}v(0,t)\cdot\frac{\partial^2}{\partial t^2}\frac{\partial^i}{\partial y^i}g(0,t)\right]\mathrm{d}t+\sum_{j=0}^{2}\left\langle\frac{\partial^j}{\partial t^j}v(y,0),\frac{\partial^j}{\partial t^j}g(y,0)\right\rangle_{W_2^3}+$$
$$+\iint_0^1\left[\frac{\partial^3}{\partial y^3}\frac{\partial^2}{\partial t^2}v(y,t)\cdot\frac{\partial^3}{\partial y^3}\frac{\partial^2}{\partial t^2}g(y,t)\right]\mathrm{d}t\,\mathrm{d}x$$

$$\|v\|_W=\sqrt{\langle v,v\rangle_W}\,,\quad v\in W(\Omega)$$

Lemma 5. Reproducing kernel function $K_{(z,s)}$ of $W(\Omega)$ is given [4]:

$$K_{(z,s)}=R_z r_s$$

Definition 6. We describe the binary space $\hat{W}(\Omega)$:

$$\hat{W}(\Omega)=\left\{v: v \text{ is completely continuous in } \Omega=[0,1]\times[0,1],\quad \frac{\partial^2 v}{\partial y \partial t}\in L^2(\Omega)\right\}$$

$$\langle v,g\rangle_{\hat{W}}=\int_0^1\left[\frac{\partial}{\partial t}v(0,t)\frac{\partial}{\partial t}g(0,t)\right]\mathrm{d}t+\langle v(y,0)g(y,0)\rangle_{H_2^1}+\int\int_0^1\left[\frac{\partial}{\partial y}\frac{\partial}{\partial t}v(y,t)\frac{\partial}{\partial y}\frac{\partial}{\partial t}g(y,t)\right]\mathrm{d}t\,\mathrm{d}y$$

and

$$\|v\|_{\hat{W}}=\sqrt{\langle v,v\rangle_{\hat{W}}},\quad v\in\hat{W}(\Omega)$$

are the inner product and norm.

Lemma 7. The kernel function $G_{(z,s)}$ of $\hat{W}(\Omega)$ is given:

$$G_{(z,s)}=T_zT_s$$

Solutions in binary reproducing kernel space

We give the solution of fractional Bloch-Torrey equation in the $W(\Omega)$. On defining the $L: W(\Omega)\to\hat{W}(\Omega)$ we get:

$$Lv=M(y,t),\ \ (y,t)\in[0,1]\times[0,1]$$

$$v(y,0)=v(0,t)=v(1,t)=0$$

Lemma 8. L is a bounded linear operator.

Now, choose a countable dense subset $\{(y_1,t_1),(y_2,t_2),\ldots\}$ in Ω and describe:

$$\varphi_i=G_{(y_1,t_1)},\quad \Psi_i=L^*\varphi_i,\quad \hat{\psi}_i=\sum_{k=1}^{i}\beta_{ik}\psi_k$$

Theorem 9. Assume $\{(y_i,t_i)\}_{i=1}^{\infty}$ is dense in Ω. Thus, $\{\hat{\psi}_i\}_{i=1}^{\infty}$ is a complete system in $W(\Omega)$, and

$$\Psi_i=LK_{(y_1t_1)}(y,t)$$

Theorem 10. If $\{(y_i,t_i)\}_{i=1}^{\infty}$ is dense in Ω, then the solution of the problem is given:

$$v=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}M(y_k,t_k)\hat{\psi}_i$$

Proof. $\{\Psi_i(y,t)\}_{i=1}^{\infty}$ is a complete system in $W(\Omega)$. Thus, we get:

$$v=\sum_{i=1}^{\infty}\left\langle v,\hat{\Psi}_i\right\rangle_W\hat{\Psi}_i=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle v,\Psi_k\right\rangle_W\hat{\Psi}_i=$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle v,L^*\varphi_k\right\rangle_W\hat{\Psi}=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lv,\varphi_k\right\rangle_{\hat{W}}\hat{\Psi}_i=$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lv,G_{(y_k,t_k)}\right\rangle_{\hat{W}}\hat{\Psi}_i=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}Lv(y_k,t_k)\hat{\Psi}_i=$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}M(y_k,t_k)\hat{\Psi}_i$$

The approximate solution v_n can be achieved by the n-term intercept of the exact solution v and

$$v_n = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} M\left(y_k, t_k\right)\hat{\Psi}_i$$

Obviously

$$\left\| v_n - v \right\|_W \to 0, \quad n \mapsto \infty$$

Numerical experiments

Example 1. We consider the following problem as an example in the reproducing kernel Hilbert space.

$${}_0^C\mathrm{D}_\tau^\alpha v(y,\tau) = \frac{1}{2}\frac{\partial^\beta v(y,\tau)}{\partial |y|^\beta} + f(y,\tau), \quad y \in (0,1), \quad \tau \in (0,1]$$

$$v(y,0) = 0, \quad y \in (0,1), \quad v(0,\tau) = 0, \quad v(1,\tau) = 0, \quad \tau \in (0,1]$$

and

$$f(y,\tau) = \left\{ \frac{\Gamma(9)}{\Gamma(9-\beta)}\left[y^{8-\beta} + (1-y)^{8-\beta}\right] - \frac{4\Gamma(8)}{\Gamma(8-\beta)}\left[y^{7-\beta} + (1-y)^{7-\beta}\right] + \right.$$
$$+ \frac{6\Gamma(7)}{\Gamma(7-\beta)}\left[y^{6-\beta} + (1-y)^{6-\beta}\right] - \frac{4\Gamma(6)}{\Gamma(6-\beta)}\left[y^{5-\beta} + (1-y)^{5-\beta}\right] +$$
$$\left. + \frac{\Gamma(5)}{\Gamma(5-\beta)}\left[y^{4-\beta} + (1-y)^{4-\beta}\right] \right\} \frac{\tau^{2+\alpha}}{4\cos\left(\frac{\beta\pi}{2}\right)} + \frac{\Gamma(3+\alpha)}{2}\tau^2 y^4 (1-y)^4$$

We obtain tab. 1 by reproducing kernel Hilbert space method.

Table 1. Maximum errors in the binary reproducing kernel Hilbert space

(α, β)	$m = 25$	$m = 36$
(0 .1, 1.1)	3.32785e–3	6.25759e–4
(0 .1, 1.1)	9.62994e–3	9.89118e–4
(0 .1, 1.1)	5.81274e–4	3.03078e–5
(0 .5, 1.9)	9.64511e–4	2.30761e–5
(0 .5, 1.9)	7.32408e–3	5.01158e–4
(0 .5, 1.9)	1.22237e–4	8.30663e–4
(0 .9, 1.5)	1.22237e–3	9.53027e–4
(0 .9, 1.5)	7.85656e–4	6.02837e–5
(0 .9, 1.5)	5.04389e–4	9.43966e–4

Conclusion

We presented the reproducing kernel Hilbert space method for solving the Fractional Bloch-Torrey Equations in this work. An example was selected to show the computational efficiency. As presented in the tab. 1 the reproducing kernel Hilbert space technique is very efficient. We used some important kernel functions in this paper.

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