NEW EXACT SOLUTIONS OF THE SPACE-TIME FRACTIONAL KdV-BURGERS AND NON-LINEAR FRACTIONAL FOAM DRAINAGE EQUATION

by

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The fractional differential equations have been studied by many authors and some effective methods for fractional calculus were appeared in literature, such as the fractional sub-equation method and the first integral method. The fractional complex transform approach is to convert the fractional differential equations into ordinary differential equations, making the solution procedure simple. Recently, the fractional complex transform has been suggested to convert fractional order differential equations with modified Riemann-Liouville derivatives into integer order differential equations, and the reduced equations can be solved by symbolic computation. The present paper investigates for the applicability and efficiency of the exp-function method on some fractional non-linear differential equations.

Key words: *fractional differential equation, exact solutions, exp-function method*

Introduction

Some scientific problems and phenomena are modeled by non-linear PDE. The phenomena of non-linear science play an important role in applied mathematics and mathematical physics. The appearance of solitary wave in nature is rather frequent, especially in fluids, plasmas, solid-state physics, condensed matter physics, optical fibers, chemical kinematics, electrical circuits, bio-genetics, elastic media, *etc.* Also the differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena. Recently, non-linear fractional partial differential equations are widely used to describe many important phenomena and dynamic processes in physics, such as electromagnetics, acoustics, viscoelasticity, electrochemistry, material science, control theory, systems identification, fractional dynamics, engineering, and other areas.

Several effective methods have been proposed so far including the fractional (G′/G)-expansion method [1-3], the fractional exp-function method [4-6], the fractional first integral method [7, 8] the fractional sub-equation method [9, 10], the fractional simplest equation method [11], and so on.

The fractional complex transform

In this section, firstly some properties and definitions of the fractional complex transform derivative are given, which are used further in this paper.

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Assume that $f: R \to R, x \to f(x)$ denote a continuous (but not necessarily differentiable) function. The fractional complex transform derivative of order α is defined by the expression [12]:

$$\mathrm{D}_x^\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]\mathrm{d}\xi, \quad \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]\mathrm{d}\xi, \quad 0 < \alpha < 1 \\ [f^{(n)}(x)]^{(\alpha-n)}, \quad n \le \alpha \le n+1, \ n \ge 1 \end{cases} \tag{1}$$

A few properties of the fractional complex transform derivative were summarized and three famous formulas of them are:

$$\mathrm{D}_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha}, \quad \gamma > 0 \tag{2}$$

$$\mathrm{D}_t^\alpha [cf(t)] = c\mathrm{D}_t^\alpha f(t), \quad c = \text{constant} \tag{3}$$

$$\mathrm{D}_t^\alpha [af(t) + bg(t)] = a\mathrm{D}_t^\alpha f(t) + b\mathrm{D}_t^\alpha g(t) \tag{4}$$

where a and b are constant.

He's exp-function method

We consider the following general non-linear foam drainage equation (FDE) of the type:

$$F(u, \mathrm{D}_t^\alpha u, \mathrm{D}_x^\beta u, \mathrm{D}_t^\alpha \mathrm{D}_t^\alpha u, \mathrm{D}_t^\alpha \mathrm{D}_x^\beta u, \mathrm{D}_x^\beta \mathrm{D}_x^\beta u, ...) = 0, \ 0 < \alpha, \quad \beta < 1 \tag{5}$$

where u is an unknown function, and F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the non-linear terms are involved. In the following, we give the main steps of the exp-function method.

Li and He [13] proposed a fractional complex transform to convert fractional differential equations into ODE, so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The fractional complex transform is of the form:

$$\begin{gathered} u(x,t) = U(\xi) \\ \xi = \frac{\tau x^\beta}{\Gamma(1+\beta)} + \frac{\lambda t^\alpha}{\Gamma(1+\alpha)} \end{gathered} \tag{6}$$

where τ and λ are non-zero arbitrary constants. By using the chain rule:

$$\begin{gathered} \mathrm{D}_t^\alpha u = \sigma_t' \frac{\mathrm{d}u}{\mathrm{d}\xi} \mathrm{D}_t^\alpha \xi \\ \mathrm{D}_x^\alpha u = \sigma_x' \frac{\mathrm{d}u}{\mathrm{d}\xi} \mathrm{D}_x^\alpha \xi \end{gathered} \tag{7}$$

where σ_t' and σ_x' are called the sigma indexes see [14-16]. Without loss of generality we can take $\sigma_t' = \sigma_x' = l$, where l is a constant.

Substituting eq. (6) with eq. (2) and eq. (7) into eq. (5), eq. (5) can be reduced into the following non-linear ODE:

$$Q(U, U', U'', U''', ...) = 0 \tag{8}$$

where the prime denotes the derivation with respect to ξ. If possible, we should integrate eq. (8) term by term one or more times.

According to exp-function method [17], we assume that the wave solution can be expressed in the following form:

$$U(\xi) = \frac{\sum_{n=-d}^{c} a_n \exp(n\xi)}{\sum_{m=-q}^{p} b_m \exp(m\xi)} \tag{9}$$

where p, q, c, and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite eq. (9) in the following equivalent form:

$$U(\xi) = \frac{a_{-d} \exp(-d\xi) + ... + a_c \exp(c\xi)}{b_{-q} \exp(-q\xi) + ... + b_p \exp(p\xi)} \tag{10}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and p, we balance the linear term of highest order of equation eq. (8) with the highest order non-linear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of eq. (8) with lowest order non-linear term. The exp-function method has been successfully applied to many kinds of non-linear differential equations [18, 19], such as high-dimensional equations [20, 21], variable-coefficient equations [22], differential-difference equations [23], and stochastic equations [24].

In this article, we will apply the exp-function method to get the exact solutions for the following space-time fractional non-linear differential equations.

The space-time fractional KdV-Burgers equation

Firstly, the non-linear space-time fractional KdV-Burgers equation [25], can be given:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon u \frac{\partial^\beta u}{\partial t^\beta} + \eta \frac{\partial^{2\beta} u}{\partial x^{2\beta}} + v \frac{\partial^{3\beta} u}{\partial x^{3\beta}} = 0, \quad t > 0, \quad 0 < \alpha, \quad \beta \le 1 \tag{11}$$

It is applied as a non-linear model of the propagation of waves on an elastic tube filled with a viscous fluid.

We introduce the following transformations:

$$u(x,t) = U(\xi), \tag{12}$$

$$\xi = \frac{cx^\beta}{\Gamma(1+\beta)} - \frac{kt^\alpha}{\Gamma(1+\alpha)}, \tag{13}$$

where c and k are non-zero constants.

Substituting eq. (13) with eq. (2) and eq. (7) into eq. (11) and once time integrating eq. (11) reduced into an ODE:

$$-kU+\varepsilon c\frac{U^2}{2}+\eta c^2U^{'}+vc^3U^{''}+\xi_0=0 \tag{14}$$

where $U'=\mathrm{d}U/\mathrm{d}\xi$ and ξ_0 is an integration constant.

Here take notice of non-linear term in eq. (14), and we can balance terms U^2 and U'' by the idea of the exp-function method [17] to determine the values of p,q,c, and d. By simple calculation, we have:

$$U^{"}=\frac{c_1\exp[(3p+c)\xi]+\cdots}{c_2\exp(4p\xi)+\cdots} \tag{15}$$

and

$$U^2=\frac{\cdots+c_3\exp(2c\xi)}{\cdots+c_4\exp(2p\xi)} \tag{16}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of exp-function in eqs. (15) and (16) we have:

$$3p+c=2c+2p \tag{17}$$

which leads to the result:

$$p=c \tag{18}$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in eq. (14)

$$U^{"}=\frac{\cdots+d_1\exp[-(3q+d)\xi]}{\cdots+d_2\exp(-4q\xi)} \tag{19}$$

and

$$U^2=\frac{d_3\exp[-2d\xi]+\cdots}{d_4\exp(-2q\xi)+\cdots} \tag{20}$$

where d_i are determined coefficients only for simplicity. From eqs. (19) and (20), we obtain:

$$-(3q+d)=-(2d+2q) \tag{21}$$

and this gives:

$$q=d \tag{22}$$

For simplicity, we set $p=c=1$ and $q=d=1$, so eq. (10) reduces to:

$$u(\xi)=\frac{a_1\exp(\xi)+a_0+a_{-1}\exp(-\xi)}{b_1\exp(\xi)+b_0+b_{-1}\exp(-\xi)} \tag{23}$$

Substituting eq. (23) into eq. (14), and by the help of symbolic computation, we obtain:

$$\frac{1}{A}[R_3\exp(3\xi)+R_2\exp(2\xi)+R_1\exp(\xi)+R_0+R_{-1}\exp(-\xi)+$$
$$+R_{-2}\exp(-2\xi)+R_{-3}\exp(-3\xi)]=0 \tag{24}$$

where

$$
\begin{aligned}
&A = [b_{-1}\exp(-\xi) + b_0 + b_1\exp(\xi)]^3 \\
&R_3 = -ka_1b_1^2 + \frac{1}{2}\varepsilon c a_1^2 b_1 \\
&R_2 = -ka_0b_1^2 - \eta c^2 a_0 b_1^2 - 2ka_1b_1b_0 + \frac{1}{2}\varepsilon c a_1^2 b_0 + vc^3 a_0 b_1^2 + \\
&\quad +\varepsilon c a_1 b_1 a_0 + \eta c^2 a_1 b_1 b_0 - vc^3 a_1 b_1 b_0 \\
&R_1 = -ka_1b_{-1}^2 - ka_{-1}b_0^2 - 2ka_0b_1b_0 + \frac{1}{2}\varepsilon c a_0^2 b_1 + \eta c^2 a_1 b_0^2 + \\
&\quad +4vc^3 a_{-1} b_1^2 + vc^3 a_1 b_0^2 - 2\eta c^2 a_{-1} b_1^2 + \frac{1}{2}\varepsilon c a_1^2 b_{-1} - 2ka_1b_1b_{-1} - \\
&\quad -\eta c^2 a_0 b_1 b_0 - vc^3 a_0 b_1 b_0 + \varepsilon c a_1 a_0 b_0 - 4vc^3 a_1 b_1 b_{-1} + \varepsilon c a_1 b_1 a_{-1} + \\
&\quad +2\eta c^2 a_1 b_1 b_{-1} \\
&R_0 = -3\eta c^2 a_{-1} b_1 b_0 + \varepsilon c a_1 b_0 a_{-1} + \frac{1}{2}\varepsilon c a_0^2 b_0 - 2ka_{-1}b_1b_0 - ka_0b_0^2 - \\
&\quad -6vc^3 a_0 b_1 b_{-1} + \varepsilon c a_0 b_1 a_{-1} + 3\eta c^2 a_1 b_0 b_{-1} + 3vc^3 a_{-1} b_0 b_1 + 3vc^3 a_1 b_0 b_{-1} - \\
&\quad -2ka_1b_0b_{-1} - 2ka_0b_1b_{-1} + \varepsilon c a_1 a_0 b_{-1} \\
&R_{-1} = -ka_{-1}b_0^2 - ka_1b_{-1}^2 + \frac{1}{2}\varepsilon c a_0^2 b_{-1} + 2\eta c^2 a_1 b_{-1}^2 - \eta c^2 a_{-1} b_0^2 + \\
&\quad +vc^3 a_{-1} b_0^2 + \frac{1}{2}\varepsilon c a_{-1}^2 b_1 - 2ka_0b_{-1}b_0 + 4vc^3 a_1 b_{-1}^2 - 2ka_{-1}b_1b_{-1} + \\
&\quad +\varepsilon c a_0 a_{-1} b_0 + \varepsilon c a_1 a_{-1} b_{-1} - 2\eta c^2 a_{-1} b_1 b_{-1} - vc^3 a_0 b_{-1} b_0 - 4vc^3 a_{-1} b_1 b_{-1} + \\
&\quad +\eta c^2 a_0 b_{-1} b_0 \\
&R_{-2} = -ka_0b_{-1}^2 + \eta c^2 a_0 b_{-1}^2 + vc^3 a_0 b_{-1}^2 - 2ka_{-1}b_0b_{-1} + \frac{1}{2}\varepsilon c a_{-1}^2 b_0 - \\
&\quad -\eta c^2 a_{-1} b_0 b_{-1} + \varepsilon c a_0 a_{-1} b_{-1} - vc^3 a_{-1} b_0 b_{-1} \\
&R_{-3} = -ka_{-1}b_{-1}^2 + \frac{1}{2}\varepsilon c a_{-1}^2 b_{-1}
\end{aligned}
\tag{25}
$$

Solving this system of algebraic equations by using MAPLE, we obtain the following results:

$$
\begin{gathered}
a_1 = 0, \quad a_0 = 0, \quad a_{-1} = -\frac{12 b_{-1}\eta^2}{25 v\varepsilon} \\
b_1 = \frac{b_0^2}{4b_{-1}}, \quad b_0 = b_0, \quad b_{-1} = b_{-1} \\
k = -\frac{6\eta^3}{125 v^2}, \quad c = \frac{\eta}{5v}
\end{gathered}
\tag{26}
$$

where b_0 and b_{-1} are free parameters. From eq. (26), substituting these results into eq. (23), we obtain:

$$u(x,t)=\frac{-\frac{12b_{-1}\eta^2}{25\nu\varepsilon}\exp\left\{-\left[\frac{\eta x^\beta}{5\nu\Gamma(1+\beta)}+\frac{6\eta^3t^\alpha}{125\nu^2\Gamma(1+\alpha)}\right]\right\}}{\frac{b_0^2}{4b_{-1}}\exp\left[\frac{\eta x^\beta}{5\nu\Gamma(1+\beta)}+\frac{6\eta^3t^\alpha}{125\nu^2\Gamma(1+\alpha)}\right]+b_0+b_{-1}\exp\left\{-\left[\frac{\eta x^\beta}{5\nu\Gamma(1+\beta)}+\frac{6\eta^3t^\alpha}{125\nu^2\Gamma(1+\alpha)}\right]\right\}} \tag{27}$$

which is the exact solution of space-time fractional KdV-Burgers equation.

The non-linear fractional FDE

The study to FDE is very significant for that the equation is a simple model of the flow of liquid through channels (Plateau borders [26]) and nodes (intersection of four channels) between the bubbles, driven by gravity and capillarity [27]. It has been studied by some scientists [28, 29]. However, as we know, the study for FDE with time-and space-fractional derivatives of this form [30]:

$$\mathrm{D}_t^\alpha u=\frac{u}{2}\mathrm{D}_x^\alpha\mathrm{D}_x^\alpha u-2u^2\mathrm{D}_x^\alpha u+(\mathrm{D}_x^\alpha u)^2 \tag{28}$$

by the exp-function method has not been investigated. Here α is the parameters standing for the order of the fractional time and space derivatives, respectively and it satisfies $0<\alpha\leq 1$ and $x>0$. When $\alpha=1$, the fractional equation reduces to FDE of the form:

$$u_t=\frac{1}{2}uu_{xx}-2u^2u_x+(u_x)^2 \tag{29}$$

We introduce the following transformations:

$$u(x,t)=U(\xi),\quad \xi=\frac{\lambda x^\alpha}{\Gamma(1+\alpha)}-\frac{ct^\alpha}{\Gamma(1+\alpha)} \tag{30}$$

where c and λ are non-zero constants.

Substituting eq. (30) with eqs. (2) and (7) into eq. (28), we can know that eq. (28) reduced into an ODE:

$$-cU'-\frac{\lambda^2}{2}UU''-2\lambda U^2U'+\lambda^2(U')^2=0 \tag{31}$$

where $U'=\mathrm{d}U/\mathrm{d}\xi$.

By the same procedure as illustrated in the Section *The space time fractional KdV-Burgers equation*, we can balance terms UU'' and U^2U' in eq. (31). We find:

$$UU''=\frac{c_1\exp[(3p+2c)\xi]+\cdots}{c_2\exp(5p\xi)+\cdots} \tag{32}$$

and

$$U^2U'=\frac{c_3\exp[(3c+p)\xi]+\cdots}{c_4\exp(4p\xi)+\cdots} \tag{33}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of exp-function in eqs. (32) and (33) we have:

$$3p+2c=3c+2p \tag{34}$$

which leads to the result:

$$p = c \tag{35}$$

Similarly, to determine values of d and q, we balance the linear term of lowest order in eq. (31):

$$UU'' = \frac{\cdots + d_1 \exp[-(3q+2d)\xi]}{\cdots + d_2 \exp(-5q\xi)} \tag{36}$$

and

$$U^2U' = \frac{\cdots + d_3 \exp[-(3d+q)\xi]}{\cdots + d_4 \exp(-4q\xi)} \tag{37}$$

where d_i are determined coefficients only for simplicity. From eqs. (36) and (37), we obtain:

$$-(3q+2d) = -(3d+2q) \tag{38}$$

and this gives:

$$q = d \tag{39}$$

For simplicity, we set $p = c = 1$ and $q = d = 1$, so eq. (10) reduces to:

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)} \tag{40}$$

Substituting eq. (40) into eq. (31), and by the help of MAPLE, we have:

$$\begin{gathered} \frac{1}{A}[R_3 \exp(3\xi) + R_2 \exp(2\xi) + R_1 \exp(\xi) + R_0 + \\ + R_{-1} \exp(-\xi) + R_{-2} \exp(-2\xi) + R_{-3} \exp(-3\xi)] = 0 \end{gathered} \tag{41}$$

where

$$\begin{gathered} A = \left[b_{-1} \exp(-\xi) + b_0 + b_1 \exp(\xi)\right]^3 \\ R_3 = ca_0 b_1^3 + 2\lambda a_1^3 b_0 + \frac{1}{2}\lambda^2 a_1^2 b_1 b_0 - ca_1 b_0 b_1^2 - 2\lambda a_1^2 a_0 b_1 - \frac{1}{2}\lambda^2 a_1 a_0 b_1^2 \\ R_2 = 3\lambda^2 a_1 a_0 b_1 b_0 - 2ca_1 b_0^2 b_1 + 2ca_0 b_1^2 b_0 - 2\lambda^2 a_1 a_{-1} b_1^2 - \\ -4\lambda a_1^2 a_{-1} b_1 + 2\lambda^2 a_1^2 b_1 b_{-1} - 2ca_1 b_{-1} b_1^2 - 4\lambda a_1 a_0^2 b_1 + 4\lambda a_1^2 a_0 b_0 - \\ -\frac{3}{2}\lambda^2 a_1^2 b_0^2 + 2ca_{-1} b_1^3 + 4\lambda a_1^3 b_{-1} - \frac{3}{2}\lambda^2 a_0^2 b_1^2 \\ R_1 = 3\lambda^2 a_1 b_0 a_{-1} b_1 + 9\lambda^2 a_1 b_{-1} a_0 b_1 - 6ca_1 b_0 b_{-1} b_1 - \\ -12\lambda a_1 a_0 a_{-1} b_1 + ca_0 b_1 b_0^2 + \frac{1}{2}\lambda^2 a_0^2 b_1 b_0 - \frac{1}{2}\lambda^2 a_0 a_1 b_0^2 + \\ +2\lambda a_1^2 a_{-1} b_0 + 2\lambda a_1 a_0^2 b_0 - \frac{11}{2}\lambda^2 a_1^2 b_{-1} b_0 + 5ca_{-1} b_1^2 b_0 - \\ -\frac{13}{2}\lambda^2 a_{-1} a_0 b_1^2 + 10\lambda a_1^2 a_0 b_{-1} + ca_0 b_1^2 b_{-1} - ca_1 b_0^3 - 2\lambda a_0^3 b_1 \end{gathered} \tag{42}$$

$$R_0 = -6\lambda^2 a_{-1}^2 b_1^2 - 4ca_1 b_0^2 b_{-1} + 5\lambda^2 a_0^2 b_1 b_{-1} + 8\lambda a_1 a_0^2 b_{-1} + 8\lambda a_1^2 a_{-1} b_{-1} -$$
$$-6\lambda^2 a_1^2 b_{-1}^2 - 8\lambda a_0^2 a_{-1} b_1 - 3\lambda^2 a_1 a_0 b_{-1} b_0 - 3\lambda^2 a_0 a_{-1} b_1 b_0 - 8\lambda a_1 a_{-1}^2 b_1 +$$
$$+12\lambda^2 a_1 b_{-1} a_{-1} b_1 - 4ca_1 b_{-1}^2 b_1 + 4ca_{-1} b_1^2 b_{-1} + 4ca_{-1} b_1 b_0^2 + \lambda^2 a_1 a_{-1} b_0^2$$

$$R_{-1} = 3\lambda^2 a_1 b_0 a_{-1} b_{-1} + 9\lambda^2 a_{-1} b_1 a_0 b_{-1} + 12\lambda a_1 a_0 a_{-1} b_{-1} +$$
$$+6ca_{-1} b_1 b_0 b_{-1} + 2\lambda a_0^3 b_{-1} + ca_{-1} b_0^3 + \frac{1}{2}\lambda^2 a_0^2 b_{-1} b_0 - \frac{1}{2}\lambda^2 a_0 a_{-1} b_0^2 -$$
$$-\frac{11}{2}\lambda^2 a_{-1}^2 b_1 b_0 - 10\lambda a_0 a_{-1}^2 b_1 - ca_0 b_1 b_{-1}^2 - 2\lambda a_1 a_{-1}^2 b_0 - \quad (42)$$
$$-2\lambda a_0^2 a_{-1} b_0 - \frac{13}{2}\lambda^2 a_1 b_{-1}^2 a_0 - ca_0 b_{-1} b_0^2 - 5ca_1 b_0 b_{-1}^2$$

$$R_{-2} = 3\lambda^2 a_0 b_0 a_{-1} b_{-1} - 2ca_1 b_{-1}^3 - \frac{3}{2}\lambda^2 a_{-1}^2 b_0^2 - 4\lambda a_{-1}^3 b_1 - \frac{3}{2}\lambda^2 a_0^2 b_{-1}^2 +$$
$$+2ca_{-1} b_1 b_{-1}^2 + 2\lambda^2 a_{-1}^2 b_1 b_{-1} + 4\lambda a_0^2 a_{-1} b_{-1} - 4\lambda a_0 a_{-1}^2 b_0 - 2ca_0 b_{-1}^2 b_0 -$$
$$-2\lambda^2 a_{-1} a_1 b_{-1}^2 + 4\lambda a_{-1}^2 a_1 b_{-1} + 2ca_{-1} b_0^2 b_{-1}$$

$$R_{-3} = -2\lambda a_{-1}^3 b_0 - ca_0 b_{-1}^3 + 2\lambda a_{-1}^2 a_0 b_{-1} - \frac{1}{2}\lambda^2 a_{-1} a_0 b_{-1}^2 + \frac{1}{2} a_{-1}^2 b_0 b_{-1} + ca_{-1} b_0 b_{-1}^2$$

Solving the previous algebraic equations by using MAPLE, we have the following results:

$$a_1 = -\frac{\lambda b_1}{2}, \quad a_0 = \frac{\lambda b_0}{2}, \quad a_{-1} = 0$$
$$b_1 = b_1, \quad b_0 = b_0, \quad b_{-1} = 0 \quad (43)$$
$$c = \frac{\lambda^3}{4}, \quad \lambda = \lambda$$

where b_0 and b_1 are free parameters. Substituting these results into eq. (40), we obtain:

$$u_1(x,t) = \frac{-\dfrac{\lambda b_1}{2}\exp\left[\dfrac{\lambda x^\alpha}{\Gamma(1+\alpha)} - \dfrac{\lambda^3 t^\alpha}{4\Gamma(1+\alpha)}\right] + \dfrac{\lambda b_0}{2}}{b_1 \exp\left[\dfrac{\lambda x^\alpha}{\Gamma(1+\alpha)} - \dfrac{\lambda^3 t^\alpha}{4\Gamma(1+\alpha)}\right] + b_0 + b_{-1}\exp\left\{-\left[\dfrac{\lambda x^\alpha}{\Gamma(1+\alpha)} - \dfrac{\lambda^3 t^\alpha}{4\Gamma(1+\alpha)}\right]\right\}} \quad (44)$$

If we take $b_1 = b_{-1} = b_0 = 1$, eq. (44) becomes:

$$u_1(x,t) = \frac{\lambda}{2}\frac{1 - \cosh\left[\dfrac{\lambda x^\alpha}{\Gamma(1+\alpha)} - \dfrac{\lambda^3 t^\alpha}{4\Gamma(1+\alpha)}\right] - \sinh\left[\dfrac{\lambda x^\alpha}{\Gamma(1+\alpha)} - \dfrac{\lambda^3 t^\alpha}{4\Gamma(1+\alpha)}\right]}{2\cosh\left\{\dfrac{\lambda x^\alpha}{\Gamma(1+\alpha)} - \dfrac{\lambda^3 t^\alpha}{4\Gamma(1+\alpha)}\right\} + 1} \quad (45)$$

which is the exact solution of the non-linear fractional FDE.

Conclusion

In this paper, the exp-function method has been successfully applied to obtain the exact solutions of the time-space fractional KdV-Burgers and the non-linear fractional FDE. Also the fractional complex transform and exp-function method are efficient and powerful method in solving a wide class of fractional order equations. As a result, many generalized and new exact solutions for them have been successfully found. Being concise and powerful, this method can also be applied to solve other fractional differential equations as long as the homogeneous balance principle is satisfied.

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