

NUMERICAL SOLUTION OF INITIAL-BOUNDARY VALUE PROBLEMS WITH INTEGRAL CONDITIONAL FOR THIRD-ORDER-DIFFERENTIAL EQUATIONS

by

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Water quality differential equation based on the theoretical bases of change is a multiparameter mathematical. When we compared with water quality measurement values, it is determined that the concentration value rate is not balanced and the two parameters, change solution is current and unique. When change conditions only one solution will not be the determinant of Jacobi matrix linear connection. Therefore, this research will help the availability in theory and uniqueness of the solution to the problem of water quality parameters.

This method provides compatibility between real data to issue water quality parameter change obtained using the equation of the estimated value of the third row and differentiative. The numerical solution of start-border value problem which is integral conditioned for third-order-differential balance and the analytical property of problem is analyzed. The application phases are shown, contribution is given theorem, some remarks about the results produced and made in the light of their theorems..

Key words: *numerical solution of start-border value problem,
water quality parameters*

Introduction

Analytical solutions to three grade differential equations are of great importance for the solution to the problem of water quality parameters but they are hardly computable [1]. In this paper, integration condition of numerical solution and analytical characteristics for three grade differential equation that is, the solution of hyperbolic differential equations with non-local boundary specifications has been investigated. Numerical solutions of hyperbolic PDE with integral conditions are still a major research area with widespread applications in engineering, physics, chemistry, and biology [2-17]. It could also be a usefull tool to express complex situation of contaminat movement across to river water downstream.

A concentration change process of contaminants in river can be explained by the theory of quality measurement system. Each cross-section of the river between the input and the output can be considered as a changing relationship between the qualities of the water [2]. According to the mass balance we can write down the basic equation of the water quality model [3-5]. The water quality model is made of an addition of a starting condition to the differential equation, and by adding the given coefficient to the model equation the concentration of polluting substances can be estimated [6-8].

In mathematics, the real problem of differential equations problems like this is called an open solution [5, 9-14]. To find the parameters, the coefficients of the water quality are calculated by using the given measurement formula. In this way, it will not be difficult to build river pollution model [15-17]. So, finding the value of the initial-boundary rivers, water quality, basic equations, and one can easily show the quality of the equation. It is possible to solve the dimension of the pollution by putting the given coefficients (parameters) into the edited equation. As we know, this is the real solution in which will have to study mathematics, general differential equations. This is the result of observation of water quality measurements to find the solution of equations for a new calculation rule.

Experimental

As water pollution parameter chemical oxygen demand (COD) data were examined to describe the water pollution by using proposed method. Water samples were collected at four sampling points of Istanbul creek discharging to Lake Sapanca. Istanbul creek is situated near the city of Sapanca and passes thorough from undisturbed forest to city, highway, and railway. Sampling point was selected before creek entering the city center (I), at the city center (II), after the highway and railway (III), and just before the lake border (IV). Measured COD values were used to express contaminant dynamic in river downstream by using differential equations technique.

Results and discussion

Third-order-differential equations, basing on aforementioned I have tried to explain, the general differential equation which is used to measure the water quality status of the first boundary value problem, numerical solution using the integral is solved. In eq. (1), the change in the entrance and the outflow and within the cross-section of any cross-section of the rivers is shown schematically, fig. 1. Because this kind of entrance-outflow system shows changes in the water quality:

$$\frac{dC_i}{dt} = \frac{Q_{i-1} C_{i-1} + Q_{ei} C_{ei} - Q_i C_i}{V_i} + r(C_i) + \sum_{h=1}^w S_{ih} \quad (1)$$

mass balance status, differential equation of the first observation point (section) according to the quality of the water [1].

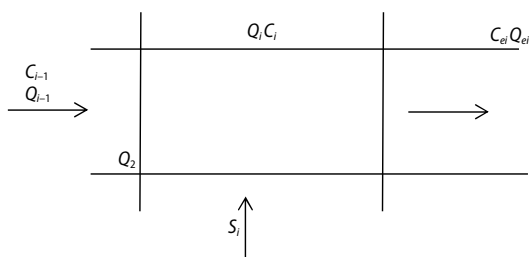


Figure 1. The figure of entrance and outflow for the water quality on the rivers

and breakdowns in the first cross-section are [11].

Let as say that soluble contaminants, conversions and reaction rate while solving the fault towel, made up of multi-parameter function:

Here C_{i-1} , C_i [ML^3] is the concentration during the entrance and the outflow in the first cross-section, Q_{i-1} , Q_i [LT^3] – the amount of stream during entrance-outflow in the first cross-section, Q_{ei} [LT^3] – the amount of out flowing water in the first cross-section [2, 3], C_{ei} [ML^{-3}] – the concentration of the specific polluting substance in the first cross-section, $r(C_i)$ [$ML^{-3}T^{-1}$] – the speed of the reaction of the soluble polluting substance in the first cross-section, and $\sum_{h=1}^w S_{ih}$ [MT^{-1}] – the total transformations

$$r(C_i) + \sum_{h=1}^w S_{ih} = \sum_{h=1}^w X_h \phi_h(t) \quad (2)$$

Here $\phi_h(t)$ is the function of the concentration C , t – the time, S – the amount of break-downs, and T – the temperature of the water, or the other factors [4]. If we put eq. (2) into eq. (1), the differential equation of the parameter changes:

$$\frac{dC_i}{dt} = \frac{Q_{i-1} C_{i-1} + Q_{ei} C_{ei} - Q_i C_i}{V_i} + \sum_{h=1}^w X_h \phi_h(t) \quad (3)$$

Here coefficient of transformations and failure. With the condition of the integral equation and the solution of the initial-boundary value problem consists of multi-parameter mass balance. The aim is now to find the solution of (X_1, X_2, \dots, X_w) .

For protecting the entrance, outflow and quality concentration of the system which is previously shown, a differential equation chance can be edited [5, 6]. The solution of differential equations or organized water quality changes if this depends on the material conditions of accuracy, given the small. This proves the existence of the problem of theoretical unity and change the balance due at the same time means that the calculation method. That means that we have to research the unity and existence of the equation solutions. On the other hand, there is more than one section of the river, and we might have to analyze them together, multi-functional function and transformed into the equation, a PDE, differential, so.

Now let us show the formula for the measurement of differential equation with two parameters, the quality of the water [7, 8]. Let us assume, that $X_1, X_2 = L_i$. That leads to $(X_1, X_2; K_1, L_1)$ and we can express eq. (1) like:

$$\frac{dc}{dt} = f(x_1, x_2, c) \quad (4)$$

In eq. (4), Jacobi matrix:

$$j = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial c} \end{bmatrix}_{m \times w} = \begin{bmatrix} -C_1 & 1 \\ -C_2 & 1 \\ \dots & \dots \\ -C_m & 1 \end{bmatrix}_{m \times 2} \quad (5)$$

According to X_1, X_2 when $\text{rank}(j) = w$, we get the single solution $X_h (h = 1, 2)$ from eq. (3). If we keep in mind, that the two equations are Jacobi matrix $|J| \neq 0$, we can get the single continuous functions $x_1, x_2, t, c, \partial c / \partial t$ from the equation system which results from these two equations systems.

On the other hand the x_i provide $(m - w)$ eq. of (5). Therefore, the existence and unity of the solution of the parameter differential equation are proved. You can see the evidence, looking at the final conditions of the two parameter water quality model $C_i(t)$, while in the first stage, any measurement based on the characteristics of that measure water quality concentration ratio is always stable. This change creates a third-order equation [9, 10]. But firstly we will create a new differential equation intended for measurements in the same time period. The unity and continuity of the solution of the previous equation, in this way, you can maintain over the same period. According to this analysis, the solution of the initial-boundary value problem

using the numerical solution of the integral conditioned, and a lot of work to investigate the problem of water quality parameters by editing the differential equation [11, 12].

Let as assume that:

$$f = \frac{dc_i}{dt} \quad (6)$$

Then

$$f = x_1\phi_1 + x_2\phi_2 + \dots + x_w\phi_w + \psi \quad (7)$$

Here is $\phi_i = \phi_i(t, c, s, T)$, and

$$\Psi = \Psi(t, c, Q, Q_e, C_e, V) = \frac{Q_{i-1}C_{i-1} + Q_{ei}C_{ei} - Q_iC_i}{V_i}$$

Let as say that these are continuous functions and that $w \leq m$ and eq. (7) are linearly linked to each other, then the Jacobi matrix has been obtained:

$$j = \begin{bmatrix} \phi_{11} & \phi_{21} & \dots & \phi_{w1} \\ \phi_{12} & \phi_{22} & \dots & \phi_{w2} \\ \dots & \dots & \dots & \dots \\ \phi_{1m} & \phi_{2m} & \dots & \phi_{wm} \end{bmatrix} \quad (8)$$

However when $\text{rank}(J) = w$, we get (x_1, x_2, \dots, x_w) , ϕ , from eq. (7).

Above the water quality's differential equation two parameters and multi-parametric changes theoretical proof and result is shown. We take the problem change in two parameters and a new method of calculation of multiple changes. Let as presume X_h linear function and examine as f from eq. (3):

$$\left(\frac{dc}{dt} - \Psi \right)_{\substack{t=t_j \\ c=c_j}} = \left[\sum_{h=1}^w X_h \phi_h \right]_{\substack{t=t_j \\ c=c_j}} \quad (9)$$

is obtained.

Here is a term that is unknown. To solve the problem of discontinuity in the original case, the original use of the first boundary value problem with an estimated value of the sampling function, but also reveals a new system of algebraic equations:

$$\left(\frac{dc}{dt} - \emptyset \right)_{\substack{t=t_j \\ c=s_j}} = \left[\sum_{h=1}^w X_h \phi_h \right]_{\substack{t=t_j \\ c=s_j}} \quad (10)$$

It is not difficult to show the estimated function of example and the linear equation. Namely, in the condition of:

$$r = \max_{1 \leq j \leq m-1} p |t_{j-1} - t_j|$$

to be small enough dc/dt and ds/dt will come closer to each other [13, 14]. Equation (10) is in linear connection according to x_1, x_2, \dots, x_w .

If we presume that:

$$a_{hj} = \phi_{ij}, \quad h = 1, 2, \dots, w, \quad j = 1, 2, \dots, m, \quad b_j = \left(\frac{dc}{dt} - \Psi \right)_{\substack{t=t_j \\ ct=s_i}}, \quad i = 1, 2, \dots, m \quad (11)$$

In this case the linear eq. (10) turns into matrix form as follows: the obtained eq. (11) may not be always a defined solution. Therefore, we should take $r = Ax - b$ and it must be:

$$\|r\|^2 = \|Ax - b\|^2 \Rightarrow \min \quad (12)$$

Consequently, if the resulting matrix equation for the small like hood method is used, the small like hood of this case the desired solution is:

$$x^* = (A^T A)^{-1} A^T B \quad (13)$$

The numeric value of the estimated value of the parameters of the difference in this (or any part of the observation point) numeric values and sample measurements, a determination is always fixed in the all the time (or balanced) to keep it higher.

Parameters of the process of change is not balanced, we can say that the equation is a vague system of equations. When m increases, the uncertainty (or complexity) increases. Uncertainty is the smallest between probability matrixes to normalize method can be estimated. The basic process to change the parameters of the resulting differential equation are studied in the case of a solution, such as linear equation matrix system concept and discontinuity technique we can find by changing shape. Meanwhile, the matrix is created with the help of the estimated equations, numerical values, allows us to reach solutions that may be required. Therefore, taking advantage of the solution of parameter changes made with the following steps. The change process aforementioned is done with the following steps:

$$N^\alpha [X, P_B \sigma] = P_B^2 (AX, B_\sigma) + \Omega_\alpha [X] \quad (14)$$

The minimum value of the function requires a fixed solution.

Here it is $P_B^2 (AX, B_\sigma) = (AX - B, AX - B)$.

If we presume it as $\Omega_\alpha [X] = (X, X)$, in this cage the solution of eq. (14) can be explained with the following:

$$A^T AZ_\alpha + Z_\alpha = A^T U \quad (15)$$

Namely,

$$Z_\alpha = (A^T A + I_\alpha)^{-1} A^T U \quad (16)$$

If $\Omega_\alpha [X]$ is balanced differential function of p order namely:

$$\Omega_\alpha [X] = \int_0^L \sum_{r=0}^p q_r(t) \left(\frac{d^r X}{dX^r} \right) dt \quad (17)$$

In this case the solution of equation is:

$$Z_\alpha = (A^T A + G_\alpha)^{-1} A^T U \quad (18)$$

Here it is the derivation of $G_\alpha, \Omega_\alpha [X]$.

If it is, the operations shown in the previous step according to the results, the parameter can be obtained from the solution. In this case, an error equation, this equation will give the

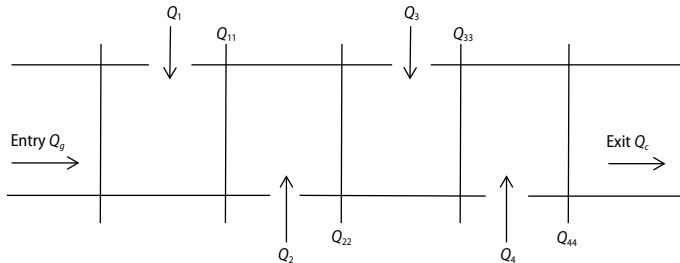


Figure 2. Scheme of the measuring points that occur between entry and exit points

of differential equation parameter changes previously shown, depending on water quality participating on the side arms of river the differential equation change models of measurement points shows the differential equation changes:

$$\frac{dC_{mi}}{dt} = \frac{Q_{i-1}C_{mi-1} + Q_{ei}C_{ei} - Q_iC_{mi}}{V_i} - K_{li}C_{mi} - K_{zi}S_i^\beta C_{mi} + KL_i - \frac{E_{q-i}}{H_i} \ln\left(\frac{S_i}{S_{io}}\right)C_{mi} \quad (19)$$

initial condition is:

$$\left. \frac{dC_{mi}}{dt} \right|_{t=0} = 0 \quad (20)$$

and water quality concentration is:

$$C_m|_{x-l_i} = C_{mi}(t) \quad (21)$$

Equations (19)-(21) by this way, the whole mass of data obtained from m group, (t_j, C_{ij}) , $j = 1, 2, \dots, m$, $i = 1, 2, \dots, n$ can be put on the model equation and desired parameters can be obtained. As against to eqs. (3) and (13) the following model equation is written:

$$b = x_1\phi_1 + x_2\phi_2 + x_3\phi_3 + x_4\phi_4 = \sum_{k=1}^4 x_k\phi_k \quad (22)$$

Here

$$b = \frac{dc}{dt} = \left. \frac{Q_{i-1}C_{i-1} + Q_{ei}C_{ei} - Q_iC_i}{V_i} \right|_{t=t_j}, \quad x_1 = -K_{11}, \quad \phi_1 = -C_{1j}, \quad x_2 = L_i,$$

$$\phi_2 = [1, 0], \quad x_3 = H_i, \quad \phi_3 = S_{ij}^\beta C_{ij}, \quad x_4 = -E_{y-i}, \quad \phi_4 = \left(\ln \frac{S_{ij}}{S_{iu}} \right) \frac{C_{ij}}{H_i}$$

On the other hand, the matrix equation can be written:

$$B = A \times X \quad (23)$$

According to the system of equations with aim to solve the problem of the time change, the matrix is known, and so it is assumed that an unknown variable is estimated. However, in this case, is possible to find the intend solution of parameters with the help of first order differential equation.

Namely,

material. If a condition equation, the parameters can be obtained from solutions.

Let as analyze the measurement data between two points of water quality parameters that are carried out. The measurement points achieved are shown in the following diagram, fig. 2.

According to the concept

$$X = (A^T)^{-1} A^T B \tag{24}$$

If the established equation systems or differential equation is a disease equation, then the solution of the parameters are:

$$X = [A^T A + G(\alpha)]^{-1} A^T B \tag{25}$$

Here α is the measured parameter and G is a corrected matrix.

Table 1. The stream's measured COD value and comparison of estimated values

Estimate point	Date	Time	Pro forma value [mg/l]	Estimate value [mg/l]	Certain variation [mg/l]	Proportional variation [mg/l]
O_2	10.2001	08:00	5.6970	5.6970	0.0000	0.0000
		09:00	9.1982	9.9292	0.7310	7.3617
		10:00	8.8333	10.2392	1.4058	13.7301
		11:00	9.6875	10.1892	0.5016	4.9233
		12:00	9.4960	9.9292	0.4332	4.3630
		13:00	9.3774	10.2392	0.8618	8.4167
		14:00	8.9980	10.1092	1.112	10.9917
		15:00	8.9252	10.2392	1.3140	12.8228
O_3	10.2001	08:00	5.6970	5.6970	0.0000	0.0000
		09:00	9.0049	10.4002	1.0740	10.2323
		10:00	9.3074	10.2792	0.9717	9.4535
		11:00	9.3638	10.2392	0.8754	8.5491
		12:00	9.1353	10.1092	0.9739	9.6339
		13:00	9.4078	10.1892	0.7814	7.6690
		14:00	9.2817	10.1092	0.8275	8.1854
		15:00	9.3960	10.0592	0.6632	6.5930
O_4	10.2001	08:00	5.4170	5.4170	0.0000	0.0000
		09:00	10.1568	10.3292	0.1723	1.6684
		10:00	10.0305	10.0192	0.0113	0.1129
		11:00	9.9898	10.2392	0.2493	2.4352
		12:00	9.8635	10.0892	0.2257	2.2371
		13:00	9.9412	9.9292	0.0121	0.1214
		14:00	9.8731	10.5892	0.7160	6.7619
		15:00	9.8245	9.9792	0.1546	1.5495
O_c	10.2001	08:00	5.3070	5.3070	0.0000	0.0000
		09:00	10.3170	10.0592	0.2579	2.5635
		10:00	10.0087	9.7992	0.2095	2.1382
		11:00	10.2275	9.7992	0.4283	4.3713
		12:00	10.0783	9.8892	0.1892	1.9127
		13:00	9.9192	9.7992	0.1200	1.2247
		14:00	10.5593	9.8492	0.7101	7.2098
		15:00	9.9533	10.0592	0.1059	1.0526
		16:00	10.2116	10.0192	0.1924	1.9204

By comparing two different solution equation parameters variation previously shown, the problem is the differences in model solutions for error. As a result, the previous equation is, at the same time I have noticed the error parameters are at the same time [15].

The solution of eq. (19), that is the water quality model equation is found:

$$C_m(t) = \frac{\phi_1}{\phi_2} - \left[\frac{\phi_1}{\phi_2} - C_m(t_{j-1}) \right] \exp(-\phi_2 \Delta t_j) \quad (26)$$

$$\phi_1 = \frac{Q_{i-1} C_{mi-1} + Q_{ei} C_{ei}}{V_i} + L_i, \quad \phi_2 = \frac{Q_i}{V_i} + K_{1i} - K_{2i} S_i^{\beta_i} + \frac{E_y}{H} \ln \frac{S_i}{S_{ui}}$$

After obtaining a cross-section of each mutual water quality parameters, the main difference between the water quality error checking is done to solve the equation. The study shows that there is an 25% error difference in this process. This error difference remains within the acceptable limits [16]. To test reality of the parameters let as have a look at the values gathered from 2001 measurement. The estimated results of the stream are given separately in tab. 1. According to the values obtained from four observation points, estimated maximum error value is 13.73%, and minimum error value is 0.1129%. Presume that the following data are obtained from four cross-sectional between 08.2001 – 10.2001. In the research the coefficients of COD are determined as follows: $K_1 = 0.2177$, $K_2 = 0.0028$, $L = -0.1707$, $E_y = 0.05447$. The obtained parameter values from stream measurement points are shown in tab. 2 and for the *Entry* point to *Exit* point cross-section 93.33% of parameters are determined as compatible [17]. Differential equations based on the theoretical basis of mathematics, multi-parameter water quality changes in water quality measurements in this study, compared with the value of the concentration ratio is not balanced, detecting a change in the two parameters the solution is unique, and this change in the present and when there is a unique problem, Jacobi determinant of the matrix is a linear relationship will not be in. Water parameters will provide the previous conditions. In conclusion, this study of the theory and the uniqueness of the solution to the problem of water quality parameters will help the situation. This solution changes the problem of water quality parameters and the results of the application of the agreement with the actual data obtained.

Table 2. The chart of measurement points which occurs between the in and out points

Years	Part	K_1	K_2	L	E_y	a	B
08.2001- 10.2001	1	0.1036	0.0002	-0.2780	-0.0311	0	0.4
	2	0.2614	0.0004	-0.1376	0.1410	1	0.2
	3	0.2438	0.0106	-0.2134	0.0425	1	0.3
	4	0.2620	0.0001	-0.0540	0.0351	0	0.3
Average		0.2177	0.0028	-0.1707	0.0547		0.3

Note: Section 1. $O_1 - O_2$, 2. $O_2 - O_3$, 3. $O_3 - O_4$, 4. $O_4 - O_5$

Conclusion

The calculation of the parameter does not change its size big-small due to other variations. This research with a real experiment, introduces a new method of distinguishing water quality parameters by setting the calculation method of the parameters without losing the problem of parameter variation.

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