AN APPLICATION OF CUBIC B-SPLINE FINITE ELEMENT METHOD FOR THE BURGERS' EQUATION

by

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It is difficult to achieve exact solution of non-linear PDE, directly. Sometimes, it is possible to convert non-linear PDE into equivalent linear PDE by applying a convenient transformation. Hence, Burgers' equation replaces with heat equation by means of the Hope-Cole transformation. In this study, Burgers' equation was converted to a set of non-linear ODE by keeping non-linear structure of Burgers' equation. In this case, solutions for each of the non-linear ODE were obtained by the help of the cubic B-spline finite element method. Model problems were considered to verify the efficiency of this method. Agreement of the solutions was shown with graphics and tables.

Key words: *Burgers` equation, cubic B-spline functions, finite element method*

Introduction

It may be regarded as a simple process to acquire the approximate solution by using the method of discretization in time which provides us to with a chance of reducing an initial boundary value problem to a system of boundary value problems which can be solved exactly or approximately [1]. In this study, approximate solution is obtained by applying well-known finite element method to each equation of the system.

Let us consider the Burgers' equation as the model problem. It is 1-D non-linear parabolic PDE:

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=\varepsilon\frac{\partial^2 U}{\partial x^2}\quad a\le x\le b\ ,\ \ t>0 \tag{1}$$

where U is the velocity, t – the time, x – the co-ordinate, and ε – the kinematics viscosity, $\varepsilon > 0$. Burgers' equation is transformed to heat equation through Hopf-Cole transformation, therefore exact solution can be obtained. Exact solution of the equation acquired was performed by Cole [2].

Burgers' equation is hyperbolic and parabolic for $\varepsilon = 0$ and $\varepsilon > 0$, respectively. Thus, it is hard to obtain a solution for small values of ε. For this reason, many researchers have used different methods to solve the Burgers' equation, such as finite element method, finite difference method, variational methods, and Adomian's decomposition method [3-8].

In this study, each of the non-linear ODE obtained by using the method of discretization in time from Burgers' equation was solved by the means of cubic B-spline finite element method. For different viscosity values at different time steps, the numerical results were com-

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pared with the exact solutions. It is obvious that approximate solution converges to the exact solution. Moreover, it was recognized that the structure of the problem for small viscosity values were preserved.

Statement of the problems

Let us consider the Burgers' eq. (1) with the homogenous boundary conditions:

$$U(0,t)=0, \quad U(1,t)=0, \quad t>0 \tag{2}$$

and the next initial conditions:

The model problem 1. The initial condition is:

$$U(x,0)=\sin\pi x \quad \text{in} \quad 0<x<1 \tag{3}$$

The exact solution of this problem was given first by Cole [2]:

$$U(x,t)=2\pi\varepsilon\frac{\sum_{n=1}^{\infty}a_n\,\mathrm{e}^{-n^2\pi^2\varepsilon t}\,n\sin n\pi x}{a_0+\sum_{n=1}^{\infty}a_n\,\mathrm{e}^{-n^2\pi^2\varepsilon t}\cos n\pi x} \tag{4}$$

where a_0 and a_n (n = 1, 2, ...) are Fourier coefficients which are given:

$$a_0=\int_0^1\exp\left(-\frac{1-\cos\pi x}{2\pi\varepsilon}\right)\mathrm{d}x$$

$$a_n=2\int_0^1\exp\left(-\frac{1-\cos\pi x}{2\pi\varepsilon}\right)\cos n\pi x\,\mathrm{d}x, \quad n\geq 1$$

The model problem 2. This problem was constructed by taking the initial condition (5) instead of the initial condition (2) in problem of eq. (1):

$$U(x,0)=4x(1-x) \quad \text{in} \quad 0<x<1 \tag{5}$$

In addition, exact solution of the *model problem 2* is also given by eq. (4), where Fourier coefficients a_0 and a_n:

$$a_0=\int_0^1\exp\left[-\frac{x^2(3-2x)}{3\varepsilon}\right]\mathrm{d}x$$

$$a_n=2\int_0^1\exp\left[-\frac{x^2(3-2x)}{3\varepsilon}\right]\cos n\pi x\,\mathrm{d}x, \quad n\geq 1$$

The method of numerical solution

Model problems by means of the method of discretization in time was converted to the following problems for the successively value of j = 1, 2, . . . , p, the functions $z_j(x)$ which are the solutions of the problems:

$$-\varepsilon z_j''(x)+z_j(x)z_j'(x)+\frac{1}{\Delta t}\left[z_j(x)-z_{j-1}(x)\right]=0 \tag{6}$$

$$z_j(0)=0, \quad z_j(1)=0 \tag{7}$$

where $z_0(x) = U(x, 0)$ [1]. The boundary value problems of eqs. (6) and (7) can be solved either exactly or numerically. Since obtaining of exact solutions is difficult as j increases, it is preferable to solve the problem numerically. The finite element method gives systematics means of generating numerical solutions to a problem formulating of the model problems. Therefore, the cubic B-spline finite element method to solve each of the problems is applied.

The weak form of eq. (6) over interval [0, 1] is given:

$$\int_0^1 v(x)\left\{-\varepsilon z_j''(x) + z_j(x)z_j'(x) + \frac{1}{\Delta t}\left[z_j(x) - z_{j-1}(x)\right]\right\}\mathrm{d}x = 0, \quad j = 1,2,\ldots,p \tag{8}$$

where $v(x)$ is weighted function. After a simple arrangement, we get:

$$\begin{aligned}\int_0^1 \varepsilon v'(x)z_j'(x)\,\mathrm{d}x + \int_0^1 v(x)z_j(x)z_j'(x)\,\mathrm{d}x + \int_0^1 \frac{1}{\Delta t}v(x)z_j(x)\,\mathrm{d}x = \\ = \left\{\varepsilon v(x)z_j'(x)\right\}\Big|_0^1 + \int_0^1 \frac{1}{\Delta t}v(x)z_{j-1}(x)dx\end{aligned} \tag{9}$$

where $j = 1, 2, \ldots, p$.

In the finite element method, the interval [0, 1] is divided into N finite elements of equal length h by the nodes x_i such that $0 = x_0 < x_1 < \ldots < x_N = 1$. Set of splines $\{\phi_{-1}, \phi_0, \ldots, \phi_{N+1}\}$ form a basis for the functions defined on [0, 1]. Cubic B-splines $\phi_m(x)$ with the required properties are defined:

$$\phi_m = \frac{1}{h^3}\begin{cases}(x - x_{m-2})^3, & [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & [x_{m-1}, x_m] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & [x_m, x_{m1}] \\ (x_{m+2} - x)^2, & [x_{m+1}, x_{m+2}] \\ 0, & \text{otherwise}\end{cases}$$

where $h = x_{m+1} - x_m$, $m = -1, 0, \ldots, N+1$ [9].

The values of $\phi_m(x)$ and its first and second derivatives $\phi_m'(x)$ and $\phi_m''(x)$ at the nodes are given by the tab. 1.

Table 1. The B-spline values at the nodes

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
$\phi_m(x)$	0	1	4	1	0
$\phi_m'(x)$	0	$3/h$	0	$-3/h$	0
$\phi_m''(x)$	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

Hence, an approximate solution $z_j^N(x)$ of the function $z_j(x)$ is given:

$$z_j^N(x) = \sum_{n=-1}^{N} c_n^j \phi_n(x) \tag{10}$$

where c_n^j are coefficients to be determined. Using eq. (10), the nodal values z_{jm} and $z'_{j,m}$ at the nodes x_m can be given:

$$\begin{aligned} z_{j,m} &= z_j(x_m) = c_{m-1}^j + 4c_m^j + c_{m+1}^j \\ z'_{j,m} &= z_j'(x_m) = \frac{3}{h}\left(-c_{m-1}^j + c_{m+1}^j\right)\end{aligned} \tag{11}$$

Because $c_j^N(x)$ should satisfy the boundary conditions (2), we get $c_{-1}^j = -(4c_0^j + c_1^j)$ and $c_{N+1}^j = -(c_{N-1}^j + 4c_N^j)$. So we have:

$$z_j^N(x) = \sum_{n=0}^{N} c_n^j \psi_n(x) \tag{12}$$

where

$$\psi_0(x) = [-4\phi_{-1}(x) + \phi_0(x)], \quad \psi_1(x) = [-\phi_{-1}(x) + \phi_1(x)], \quad \psi_m(x) = \phi_m(x), \; (m = 2,3,...,N-2)$$

$$\psi_{N-1} = [\phi_{N-1}(x) - \phi_{N+1}(x)], \quad \psi_N = [\phi_N(x) - 4\phi_{N+1}(x)]$$

Thus the unknown c_m^j (m = 0, 1, ..., N) must be obtained.

Let us construct approximate solution of eq. (9) by using Galerkin method. In this method, the weighted function $v(x)$ is taken as $v_m(x) = \psi_m(x)$, (m = 0, 1, ...,N). Substituting eqs. (11) and (12) into eq. (9), we get:

$$\varepsilon \sum_{n=0}^{N} \left(\int_0^1 \psi_k' \psi_n' \,\mathrm{d}x \right) c_n^j + \sum_{n=0}^{N} \sum_{m=0}^{N} \left(\int_0^1 \psi_k \psi_n \psi_m' \,\mathrm{d}x \right) c_m^j c_n^j + \frac{1}{\Delta t} \sum_{n=0}^{N} \left(\int_0^1 \psi_k \psi_n \,\mathrm{d}x \right) c_n^j = \frac{1}{\Delta t} \int_0^1 \psi_k z_{j-1} \mathrm{d}x \tag{13}$$

where j = 1, 2, ..., p and k = 0, 1, ...,N. After some operations, eq. (13) becomes in the matrix form:

$$\varepsilon A c^j + B(c^j) c^j + \frac{1}{\Delta t} C c^j = \frac{1}{\Delta t} D^{j-1} \tag{14}$$

where $c^j = (c_0^j, c_1^j, c_{N-1}^j)^T$, j = 1, 2, ..., p. Equation (14) is a $(N+1)\times(N+1)$ non-linear equations system for each j. The matrices A, B, and C are $(N+1)\times(N+1)$ 7-diagonal matrices and m^{th} row of these matrices have the form:

$$A : \frac{1}{10h}\left(-3, -72, -45, 240, -45, -72, -3\right)$$

$$B : \frac{1}{840}[(-5, -21, 21, 5, 0, 0, 0)c_m^j, (-108, -1944, 0, 1944, 108, 0, 0)c_m^j,$$
$$(-129, -8130, -17841, 17841, 8130, 129, 0)c_m^j,$$
$$(-10, -3888, -35682, 0, 35682, 3888, 10)c_m^j,$$
$$(0, -129, -8130, -17841, 17841, 8130, 129)c_m^j, (0, 0, -108, -1944, 0, 1944, 108)c_m^j,$$
$$(0, 0, 0, -5, -21, 21, 5)c_m^j]$$

where $c_m^j = (c_{m-4}^j, c_{m-3}^j, c_{m-2}^j, c_{m-1}^j, c_m^j, c_{m+1}^j, c_{m+2}^j)^T$,

$$C : \frac{h}{140}\left(1, 120, 1191, 2416, 1191, 120, 1\right)$$

where $c_m^j = (c_{m-4}^j, c_{m-3}^j, c_{m-2}^j, c_{m-1}^j, c_m^j, c_{m+1}^j, c_{m+2}^j)^T$. The D^{j-1} is an $N+1$ column vector. D^0 is calculated by using composite Simpson's rule [10] for both of problem and m^{th} row of D^{j-1} is given:

$$D^{j-1} : \frac{h}{140}\left(c_{m-4}^{j-1}, 120c_{m-3}^{j-1}, 1191c_{m-2}^{j-1}, 2416c_{m-1}^{j-1}, 1191c_m^{j-1}, 120c_{m+1}^{j-1}, c_{m+2}^{j-1}\right), \quad j \geq 2$$

for each problem.

The numerical results and conclusion

The non-linear systems, eq. (14), were solved for each j-value by using Newton method which rapidly converges to solution. Hence, Jacobian matrix in the algorithm of the Newton method is 7-diagonal matrix because of choosing the basis functions in the finite element method as cubic B-spline functions. System obtained for each j-value has been solved with a direct method. In the right hand side of the first system, D^0 was calculated by using composite Simpson's rule.

We consider the two model problems to show the efficiency of the given method. The numerical solutions obtained by this way have been compared with the exact solutions for various ε values at different times. The number of both divisions of the interval [0, T] and basis elements was increased to appreciate the numerical solutions. It is seen that as p is increasing numerical solution converge to the exact solution, tab. 2. Similarly, as the number of basic functions increase the numerical solutions converge to exact solution, too tab. 3.

Table 2. Numerical solutions and the exact solution of *problem 1* at various values of Δt for $\varepsilon = 1$, $N = 10$ at $t = 0.1$

x	Numerical solutions			Exact solution
	$\Delta t = 0.0001$	$\Delta t = 0.00005$	$\Delta t = 0.00001$	
0.1	0.10962	0. 10956	0.10954	0.10954
0.2	0.20987	0. 20982	0.20979	0.20979
0.3	0. 29198	0. 29192	0.29190	0.29190
0.4	0. 34797	0. 34791	0.34792	0.34792
0.5	0. 37158	0. 37154	0.37157	0.37158
0.6	0. 35891	0. 35892	0.35902	0.35905
0.7	0. 30977	0. 30978	0.30989	0.30991
0.8	0. 22732	0. 22748	0.22778	0.22782
0.9	0. 12060	0.12058	0.12065	0.12069

Table 3. Numerical solutions and the exact solution of *problem 1* with different number of basis elements Δt for $\varepsilon = 1$, $\Delta t = 0.0001$ at $t = 0.1$

x	Numerical solutions			Exact solution
	$N = 10$	$N = 20$	$N = 40$	
0.1	0.22346	0.22345	0.22345	0.22345
0.2	0.43583	0.43581	0.43580	0.43580
0.3	0.62515	0.62513	0.62512	0.62512
0.4	0.77773	0.77768	0.77772	0.77772
0.5	0.87729	0.87728	0.87728	0.87728
0.6	0.90420	0.90422	0.90424	0.90425
0.7	0.83694	0.83687	0.83690	0.83692
0.8	0.65722	0.65725	0.65729	0.65731
0.9	0.36584	0.36545	0.36573	0.36575

The numerical solutions obtained for $0.0001 \leq \varepsilon \leq 1$ at different time steps were given by figs. 1-6 and tabs. 4-6. Obviously, it was found that the mathematical structure of problems for the obtained numerical solutions in very small values of viscosity ($\varepsilon < 0.01$) at different time steps did not change, figs. 4 and 5.

Table 4. Numerical solutions and the exact solution of *problem 1* at various values of ε and N = 20, Δt = 0.0001 at different times

x	t	ε = 0.1		ε = 0.01	
		Numerical solution	Exact solution	Numerical solution	Exact solution
0.25	0.4	0.30891	0.30889	0.34195	0.34191
	0.6	0.24075	0.24074	0.26898	0.26896
	0.8	0.19568	0.19568	0.22151	0.22148
	1.0	0.16257	0.16256	0.18821	0.18819
	3.0	0.02719	0.02720	0.07512	0.07511
0.5	0.4	0.56964	0.56963	0.66075	0.66071
	0.6	0.44721	0.44721	0.52946	0.52942
	0.8	0.35923	0.35924	0.43918	0.43914
	1.0	0.29191	0.29192	0.37446	0.37442
	3.0	0.04019	0.04021	0.15019	0.15018
0.75	0.4	0.62538	0.62544	0.91022	0.91026
	0.6	0.48717	0.48721	0.76726	0.76724
	0.8	0.37389	0.37392	0.64744	0.64740
	1.0	0.28745	0.28747	0.55607	0.55605
	3.0	0.02975	0.02977	0.22483	0.22481

Table 5. Comparison of the numerical solutions of *problem 2* with the both result from [6] and the exact solutions for ε =1 at different times

x	t	Numerical solutions		Exact solution
		N = 80, Δt = 0.00001 [6]	N = 20, Δt = 0.0001 Present	
0.25	0.10	0.26149	0.26147	0.26148
	0.15	0.16148	0.16147	0.16148
	0.20	0.09947	0.09946	0.09947
	0.25	0.06109	0.06107	0.06108
0.5	0.10	0.38343	0.38340	0.38342
	0.15	0.23406	0.23403	0.23406
	0.20	0.14289	0.14285	0.14289
	0.25	0.08723	0.08720	0.08723
0.75	0.10	0.28157	0.28155	0.28157
	0.15	0.16974	0.16973	0.16974
	0.20	0.10266	0.10263	0.10266
	0.25	0.06229	0.06226	0.06229

The most fundamental difference of this study from the literature is that although larger Δt and less number of base element are selected more good results have been obtained.

This implies that the method used is very economics. The method of solution presented provides high accuracy. Finally, it can be used to solve Burgers'-like equations.

Table 6. Comparison of the numerical solutions of *problem 2* with the both result from [6] and the exact solutions for $\varepsilon = 0.1$ at different times

x	t	Numerical solutions		Exact solution
		$N = 80$, $\Delta t = 0.00001$ [6]	$N = 20$, $\Delta t = 0.0001$ Present	
0.25	0.4	0.31759	0.31754	0.31752
	0.8	0.24618	0.24615	0.24614
	0.8	0.19959	0.19956	0.19956
	1.0	0.16562	0.16561	0.16560
	3.0	0.02776	0.02775	0.02775
0.5	0.4	0.58460	0.58455	0.58454
	0.8	0.45803	0.45798	0.45798
	0.8	0.36744	0.36740	0.36740
	1.0	0.29838	0.29833	0.29834
	3.0	0.04107	0.04105	0.04106
0.75	0.4	0.64559	0.64557	0.64562
	0.8	0.50269	0.50263	0.50268
	0.8	0.38536	0.38530	0.38534
	1.0	0.29588	0.29582	0.29586
	3.0	0.03044	0.03042	0.03044

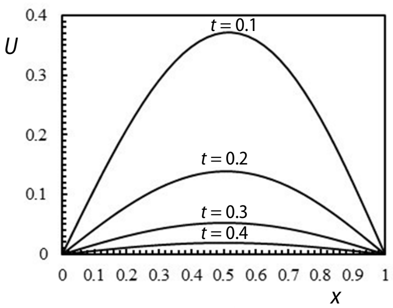

Figure 1. Solutions of *problem 1* at different times for $\varepsilon = 1$, $N = 20$, and $\Delta t = 0.0001$

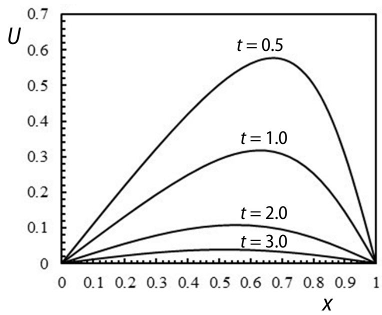

Figure 2. Solutions of *problem 1* at different times for $\varepsilon = 0.1$, $N = 20$, and $\Delta t = 0.0001$

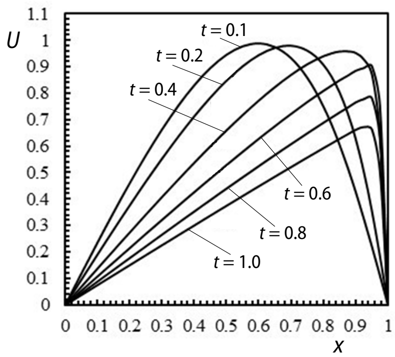

Figure 3. Solutions of *problem 1* at different times for $\varepsilon = 0.01$, $N = 20$, and $\Delta t = 0.0001$

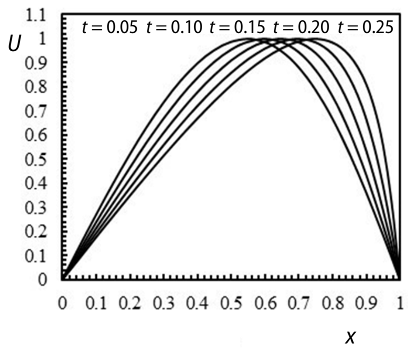

Figure 4. Solutions of *problem 1* at different times for $\varepsilon = 0.001$, $N = 20$, and $\Delta t = 0.0001$

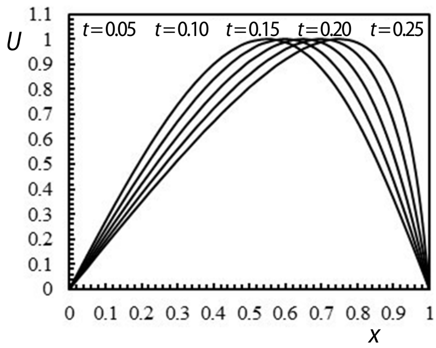

Figure 5. Solutions of *problem 1* at different times for $\varepsilon = 0.0001$, $N = 20$, and $\Delta t = 0.0001$

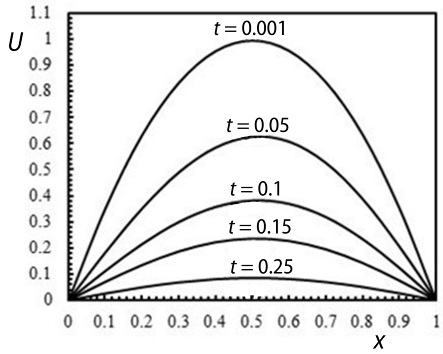

Figure 6. Solutions of *problem 2* at different times for $\varepsilon = 1$, $N = 20$, and $\Delta t = 0.0001$

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