NUMERICAL SOLUTIONS OF THE FRACTIONAL KdV-BURGERS-KURAMOTO EQUATION

by

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Non-linear terms of the time-fractional KdV-Burgers-Kuramoto equation are linearized using by some linearization techniques. Numerical solutions of this equation are obtained with the help of the finite difference methods. Numerical solutions and corresponding analytical solutions are compared. The L_2 *and* L_∞ *error norms are computed. Stability of given method is investigated by using the Von Neumann stability analysis.*

Key words: *time-fractional KdV-Burgers-Kuramoto equation, finite difference method, stability*

Introduction

The birth of fractional calculus goes as Leibniz and Newton's differential calculus. The notion of fractional-order derivative of non-integer order was firstly introduced by Leibniz:

$$\frac{\mathrm{d}^n \mathrm{e}^{mx}}{\mathrm{d}x^n} = m^n \mathrm{e}^{mx} \tag{1}$$

where n is non-integer value (cf. [1]).

Later, fractional PDE drew attention of many mathematicians and have also shown an increasing development (cf. [2-10]).

Time-fractional KdV-Burgers-Kuramoto equation is one of the important fractional difference equation which is defined by the following form:

$$\frac{\partial^\gamma u}{\partial t^\gamma} + u\frac{\partial u}{\partial x} - \lambda_1 \frac{\partial^2 u}{\partial x^2} + \lambda_2 \frac{\partial^3 u}{\partial x^3} + \lambda_3 \frac{\partial^4 u}{\partial x^4} = 0 \tag{2}$$

with initial condition

$$u(x,0) = f\left(x\right) \tag{3}$$

and with boundary conditions

$$u(a,t) = \beta_1, \quad u(b,t) = \beta_2, \quad t \ge t_0, \tag{4}$$

where γ is the order of the fractional time derivative and λ_1, λ_2, and $\lambda_3 \ge 0$ are parameters characterizing instability, dispersion and dissipation, respectively [11].

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Time-fractional KdV-Burgers-Kuramoto equation is an important model to describe physical phenomena on the move of turbulence and other instability process. It can also be used to define a long waves on a viscous fluid flowing down along an inclined plane [12], unstable drift waves in plasma [13], turbulent cascade model in a barotropic atmosphere [14].

In this paper, we study the finite difference method to obtain numerical solutions of time-fractional KdV-Burgers-Kuramoto equation. The effectiveness of the proposed method is tested using a numerical example and the stability analysis is investigated.

Finite difference methods

In this section, we shall recall some basic facts about finite difference methods.

Firstly, we shall define a set of grid points in the domain D to obtain a numerical solution to eq. (2) using finite difference methods as follows.

Let us choose a state step size $\Delta x(h) = (b-a)/N$ (N is an integer), a time step size Δt, draw a set of horizontal and vertical line across D, and get all intersection points (x_j, t_n) or simply (j,n) where $x_j = a + j\Delta x$, $j = 0,1,2,\cdots,N$, and $t_n = n\Delta t$, $n = 0,1,\ldots,M$. If $D = [a,b]\times[0,T]$, then we can choose $\Delta t = T/M$ (M is an integer) and $t_n = n\Delta t$, $n = 0,1,\ldots,M$.

Putting appropriate finite difference approximation in eq. (2), it can be seen that solution of eq. (2) reduced to solution problem of algebraic differential systems of linear and non-linear equations consisting of finite difference equation.

However, it is known that non-linear system of equations can not be solved directly. Therefore, we shall use two linearization techniques for non-linear terms existing in the eq. (2).

Linearization 1

Firstly, putting the Caputo fractional derivative approximation for $\partial^\gamma u/\partial t^\gamma$ $(0<\gamma\le 1)$ and usual finite difference approximations at the nodal point $(m, n+1)$ which are given, respectively, by the following forms [10, 15]:

$$\frac{\partial^\gamma u}{\partial t^\gamma} \simeq \begin{cases} \dfrac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\displaystyle\sum_{k=0}^{n}\left(u_m^{n+1-k}-u_m^{n-k}\right)\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right], & n\ge 1 \\ \dfrac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\left(u_m^1-u_m^0\right), & n=0 \end{cases}$$

$$U_{xx} \simeq \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \tag{5}$$

$$U_{xxx} \simeq \frac{1}{2h^3}\left(u_{m+2}^n - 2u_{m+1}^n + 2u_{m-1}^n - u_{m-2}^n\right)$$

$$U_{xxxx} \simeq \frac{u_{m+2}^n - 4u_{m+1}^n + 6u_m^n - 4u_{m-1}^n - u_{m-2}^n}{h^4}$$

in eq. (2) and later applying the following linearization technique defined:

$$uu_x \simeq \left(u_m^n\right)\left(\frac{u_{m+1}^n - u_{m-1}^n}{2h}\right)$$

for the non-linear term uu_x in eq. (2) [16], consequently we have the following system of algebraic equation:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left(u_m^{n+1-k}-u_m^{n-k}\right)\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]=$$
$$=-(u)_m^n\left(\frac{u_{m+1}^n-u_{m-1}^n}{2h}\right)+\lambda_1\left(\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}\right)-$$
$$-\lambda_2\left(\frac{u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n}{2h^3}\right)-\lambda_3\left(\frac{u_{m+2}^n-4u_{m+1}^n+6u_m^n-4u_{m-1}^n-u_{m-2}^n}{h^4}\right) \tag{6}$$

Linearization 2

Firstly, putting eq. (5) in eq. (2) and later applying the following linearization technique defined:

$$uu_x \simeq \left(\frac{u_m^n+u_{m+1}^n}{2}\right)\frac{1}{2h}\left(u_{m+1}^n-u_{m-1}^n\right) \tag{7}$$

in eq. (2), then we have the following system of algebraic equation:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left(u_m^{n+1-k}-u_m^{n-k}\right)\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]=$$
$$=-\frac{\left(u_m^n+u_{m+1}^n\right)\left(u_{m+1}^n-u_{m-1}^n\right)}{4h}+\lambda_1\left(\frac{u_{m+1}^n-2u_m^n+u_{m-1}^n}{h^2}\right)-$$
$$-\lambda_2\left(\frac{u_{m+2}^n-2u_{m+1}^n+2u_{m-1}^n-u_{m-2}^n}{2h^3}\right)-\lambda_3\left(\frac{u_{m+2}^n-4u_{m+1}^n+6u_m^n-4u_{m-1}^n+u_{m-2}^n}{h^4}\right) \tag{8}$$

Stability analysis

In this subsection, stability of the finite difference method is investigated with the help of Von-Neumann analysis.

Taking into consideration the Von-Neumann stability analysis, the growth factor of a typical Fourier mode is defined:

$$u_m^n=\xi^n \mathrm{e}^{im\phi} \tag{9}$$

where $i=\sqrt{-1}$ and ξ^n denotes the amplication factor.

To investigate the stability of the numerical scheme, the non-linear term uu_x in the modified Burgers' equation has been linearized by making the quantity u to a local constant. Thus the non-linear term in the equation converts into $\hat{u}u_x$ and in that case the eq. (2) turns into:

$$\frac{\partial^\gamma u}{\partial t^\gamma}+\hat{u}\frac{\partial u}{\partial x}-\lambda_1\frac{\partial^2 u}{\partial x^2}+\lambda_2\frac{\partial^3 u}{\partial x^3}+\lambda_3\frac{\partial^4 u}{\partial x^4}=0 \tag{10}$$

Substituting the Fourier mode of eq. (9) into the recurrence relationship of eq. (6), we get:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left(\xi^{n-k+1}\mathrm{e}^{im\phi}-\xi^{n-k}\mathrm{e}^{im\phi}\right)\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]+$$

$$+\hat{u}\left[\frac{\xi^n e^{i(m+1)\phi}-\xi^n e^{i(m-1)\phi}}{2h}\right]-\lambda_1\left[\frac{\xi^n e^{i(m+1)\phi}-2\xi^n e^{im\phi}+\xi^n e^{i(m-1)\phi}}{h^2}\right]+$$
$$+\lambda_2\left[\frac{\xi^n e^{i(m+2)\phi}-2\xi^n e^{i(m+1)\phi}+2\xi^n e^{i(m-1)\phi}-\xi^n e^{i(m-2)\phi}}{2h^3}\right]+$$
$$+\lambda_3\left[\frac{\xi^n e^{i(m+2)\phi}-4\xi^n e^{i(m+1)\phi}+6\xi^n e^{im\phi}-4\xi^n e^{i(m-1)\phi}+\xi^n e^{i(m-2)\phi}}{h^4}\right]=0 \quad (11)$$

Let $\xi^{n+1}=\zeta\xi^n$ and assume that $\zeta=\zeta(\phi)$ is independent of time. Then, we obtain the following:

$$\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left(\xi^{-k+1}-\xi^{-k}\right)\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]+\hat{u}\left(\frac{e^{i\phi}-e^{-i\phi}}{2h}\right)-$$
$$-\lambda_1\left(\frac{e^{i\phi}-2+e^{-i\phi}}{h^2}\right)+\lambda_2\left(\frac{e^{2i\phi}-2e^{i\phi}+2e^{-i\phi}-e^{-2i\phi}}{2h^3}\right)+$$
$$+\lambda_3\left(\frac{e^{2i\phi}-4e^{i\phi}+6-4e^{-i\phi}+e^{-2i\phi}}{h^4}\right)=0 \quad (12)$$

Then we easily obtain the following expression:

$$\xi=\frac{i\left(-\frac{\hat{u}}{h}+2\frac{\lambda_2}{h^3}\right)\sin(\phi)+\frac{2\lambda_1}{h^2}[\cos(\gamma)-1]-\frac{4\lambda_3}{h^4}[\cos^2(\gamma)-2\cos(\gamma)+1]}{\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left(\xi^{-k}-\xi^{-k-1}\right)\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]}$$

For more details, we refer to [17].

Hence, we get:

$$\xi=\frac{X_1+iX_2}{Y_1} \quad (13)$$

where

$$X_1=\frac{2\lambda_1}{h^2}[\cos(\gamma)-1]-\frac{4\lambda_3}{h^4}[\cos^2(\gamma)-2\cos(\gamma)+1]$$
$$X_2=\left(-\frac{\hat{u}}{h}+2\frac{\lambda_2}{h^3}\right)\sin(\phi)$$
$$Y_1=\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n}\left(\xi^{-k}-\xi^{-k-1}\right)\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]$$

According to the Fourier stability, for the given scheme to be stable, the condition $|\xi|\le 1$ must be satisfied. This implies that if the following inequality:

$$|\zeta|^2=\frac{X_1^2+X_2^2}{Y_1^2}\le 1 \quad (14)$$

is provided then the schema become unconditionally stable:

Numerical examples and results

In this subsection, numerical results of the equation have been obtained for the test problem used in the present study.

To show how accurate the results, both the error norm L_2 given:

$$L_2 = \left\| U^{\text{exact}} - U_N \right\|_2 = \sqrt{h \sum_{J=0}^{N} \left| U_j^{\text{exact}} - \left(U_N \right)_j \right|^2}$$

and the error norm L_∞ given:

$$L_\infty = \left\| U^{\text{exact}} - U_N \right\|_\infty = \max_j \left| U_j^{\text{exact}} - \left(U_N \right)_j \right|$$

are going to be computed and presented.

Test problem

The analytical solution of the fractional KdV- Burgers-Kuramoto is given:

$$u(x,t) = v - \frac{45\sqrt{11}}{153\sqrt{19}} \left(a + a^{-1} \right) + \frac{165\sqrt{11}}{152\sqrt{19}} \left(a^3 + a^{-3} \right), \quad t \ge t_0, \quad 0 \le x \le 1 \tag{15}$$

where

$$a = \tanh \left[\frac{\sqrt{11}}{4\sqrt{19}} \left(x - vt - b \right) \right], \quad v = 1, \quad b = 1.$$

In our computations, three different linearization techniques have been applied for the numerical solution of the test problem. The error norms L_2 and L_∞ are computed for $\Delta t = h = 0.0002$. In tab. 1, the error norms L_2 and L_∞ obtained using the linearization techniques are compared. In fig. 1, solution for linearization 1, solution linearization 2 and exact solution for $h = 0.0002$ are compared.

Table 1. Comparison of error norms L_2 and L_∞ using the linearization techniques at $\gamma = 0.2$ for $h = 0.0002$, $0 \le x \le 1$

	Numerical solution		Exact solution
x	Linearization 1	Linearization 2	
0	17.5999	17.6	17.5968
0.01	17.5844	17.5845	17.5736
0.02	17.5421	17.5422	17.5504
0.03	17.576	17.5761	17.5272
0.04	17.5517	17.5517	17.5041
0.05	17.4643	17.5275	17.4811
$L_2 \cdot 10^2$	0.00372837	0.00367668	
$L_\infty \cdot 10^2$	0.0580657	0.0581451	

Conclusion

Numerical solutions of the fractional KdV-Burgers-Kuramoto equation are obtained by using finite difference methods with two different linearization techniques. The computational efficiency and effectiveness of proposed method are tested on a problem. The error norms L_2 and L_∞ are computed and presented. The obtained results show that the error norms are sufficiently small during all computer runs. Considering the tables, it is obvious that the obtained results using linearization 1 is better than obtained results using other linearization. It is shown that the present method is a particularly successful numerical scheme to solve the fractional KdV-Burgers-Kuramoto equation.

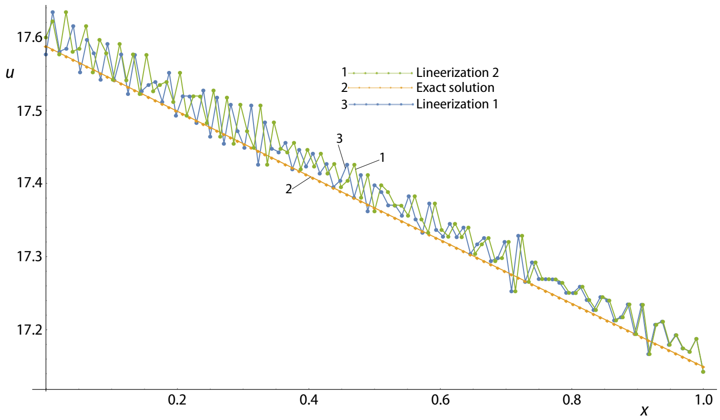

Figure 1. Comparison of exact solution and numerical solutions which are obtained with the help of linearization techniques at $\gamma = 0.2$ for $h = 0.0002,\ 0 \le x \le 1$

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