CLOSED FORM TRAVELING WAVE SOLUTIONS OF NON-LINEAR FRACTIONAL EVOLUTION EQUATIONS THROUGH THE MODIFIED SIMPLE EQUATION METHOD

by

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Original scientific paper
https://doi.org/10.2298/TSCI170613097A

In this article, the modified simple equation (MSE) method is introduced to examine the closed form wave solutions of the fractional non-linear Cahn-Allen equation and of the fractional generalized reaction Duffing equation. The fractional derivatives are delineated in the sense of Jumarie's modified Riemann-Liouville derivative. A fractional complex transformation is used to transform the fractional-order PDE into integer order ODE. The reduced equations are then examined by using the MSE method and some new and further general solutions of these equations are successfully established. The approach of this method is simple, standard and the obtained solutions are highly encouraging. It is also powerful, reliable and effective.

Key words: *fractional differential equations, fractional Cahn-Allen equation, fractional generalized Duffing equation, fractional complex transformation, modified simple equation method*

Introduction

Fractional differential equations (FDE) are the generalization of classical differential equations of integral order. In the recent years, the study on non-linear FDE has been a lucrative area of research, because of the development of the theory of fractional calculus as well as its applications in various branches of science and engineering, including transmission of electrical current in cables, electromagnetic theory, signal processing, biology, chemical reaction, diffusion, probability, fractional dynamics [1], *etc.* Oldman and Spanier [2] first considered the FDE arising in diffusion problems. Thus, for better comprehension the mechanisms of the intricate non-linear phenomena as well as further applying them to the real world problems, the investigation of solutions of FDE is very much important. In the last several years, some effective and useful methods have been established, as for instance, the fractional sub-equation method [3], the simplest equation method [4], the first integral method [5, 6], the exp-function method [7, 8], the functional variable method [9], the (G'/G)-expansion method [10], the trial equation method [11, 12], the Adomian decomposition method [13], the variational method [14], the extended algebraic method [15, 16], the fractional direct algebraic method [17], the Darcy law method [18], the modified extended direct algebraic method [19], the generalized

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projective-Riccati equation method and the Riccati-Bernoulli sub-ODE method [20], Jacobi elliptic functions method [21], the exponential rational function method [22], the extended trial equation method [23], the homotopy perturbation method [24], the meshless method of lines [25], the He's polynomials method [26], the variational iteration method [27], *etc.*

The MSE method [28, 29] is an easily usable effective method to examine closed form solutions to non-linear evolution equations in science and engineering. The objective of this article is to investigate the applicability and efficiency of the MSE method to extract closed form wave solutions to the fractional order non-linear PDE. To this end, the method is applied to some fractional PDE, namely, the fractional non-linear Cahn-Allen equation and the fractional generalized reaction Duffing equation.

Jumarie's modified Riemann-Liouville derivative

There are several definitions of fractional order derivative of order $\alpha > 0$ [2]. Among them, the Riemann-Liouville and Caputo [30] definitions are frequently used. Jumarie [30] introduced a modification of Riemann-Liouville derivative which is used in this article.

Suppose that $f: R \to R$ denotes a continuous but not necessarily be a first order differentiable function. By means of the Riemann-Liouville fractional integral:

$$\mathrm{D}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\,\mathrm{d}\xi, \quad 0 < \alpha < 1 \tag{1}$$

Jumarie modified Riemann-Liouville derivative of order α is defined by [15]:

$$\mathrm{D}_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^t (t-\xi)^{-\alpha-1} \left[f(\xi) - f(0)\right] \mathrm{d}\xi, & \alpha < 0 \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_0^t (t-\xi)^{-\alpha} \left[f(\xi) - f(0)\right] \mathrm{d}\xi, & 0 < \alpha < 1 \\ \left[f^n(t)\right]^{\alpha-n}, & n \le \alpha \le n+1, \quad n > 1 \end{cases} \tag{2}$$

which has advantages over the original definition is that the α-order derivative of a constant is zero. Furthermore, some of the useful properties of the Jumarie modified Riemann-Liouville derivative are provided:

$$\mathrm{D}_t^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \quad \beta > 0 \tag{3}$$

$$\mathrm{D}_t^\alpha \left[f(t)g(t)\right] = f(t)\mathrm{D}_t^\alpha \left[g(t)\right] + g(t)\mathrm{D}_t^\alpha \left[f(t)\right] \tag{4}$$

$$\mathrm{D}_t^\alpha \left\{h\left[f(t)\right]\right\} = h'_f \left[h(t)\right] \mathrm{D}_t^\alpha \left[f(t)\right] \tag{5}$$

The eqs. (3)-(5) will be used to obtain solutions of FDE in the following sections.

The fractional complex transform

We consider the time FDE with independent variable $x = (x_1, x_2, x_3, \cdots, x_n, t)$ and a dependent variable u in the following:

$$F(u, \mathrm{D}_t^\alpha u, u_{x_1}, u_{x_2}, u_{x_3}, \mathrm{D}_t^{2\alpha} u, u_{x_1,x_1}, u_{x_2,x_2}, u_{x_3,x_3}, \cdots) = 0 \tag{6}$$

Consider the fractional complex transformation:

$$u(\xi) = u(x_1, x_2, x_3, \cdots, x_n, t), \quad \xi = \sum_{i=1}^{n} k_i x_i - \frac{\omega t^{\alpha}}{\Gamma(\alpha+1)} \tag{7}$$

where k_i and ω are constants to be evaluated later. Similarly, we can suppose the space FDE with independent variable $x = (x_1, x_2, x_3, \cdots, x_n, t)$ and dependent variable u in this way:

$$G(u, u_t, D_x^{\beta} u, u_{x_1}, u_{x_2}, u_{x_3}, D_x^{2\beta} u, u_{x_1, x_1}, u_{x_2, x_2}, u_{x_3, x_3}, \cdots) = 0 \tag{8}$$

In this case, we make use the following fractional complex transformation:

$$u(\xi) = u(x_1, x_2, x_3, \cdots, x_n, t), \quad \xi = \frac{k x^{\beta}}{\Gamma(1+\beta)} + \sum_{i=1}^{n} k_i x_i - \omega t \tag{9}$$

where k_i and ω are arbitrary constants to be find out.

Using fractional complex transformations of eqs. (7) and (9), the FDE (6) and (8) will be transformed into a non-linear ODE in the subsequent structure:

$$H(u, -\omega u', k u', \omega^2 u'', k^2 u'', \cdots) = 0 \tag{10}$$

where primes denote ordinary derivatives with respect to ξ.

The method

Let us consider the non-linear fractional PDE of the form:

$$F(u, D_t^{\alpha} u, D_x^{\beta} u, D_t^{2\alpha} u, D_t^{\alpha} D_x^{\beta} u, D_x^{2\beta} u, \cdots) = 0, \quad 0 < \alpha, \beta < 1 \tag{11}$$

where u is an indeterminate, $D_t^{\alpha} u$, $D_x^{\beta} u$, are Jumarie's modified Riemann-Liouville derivatives of u, and F is a polynomial of u and its fractional derivatives. The main steps of the MSE method are discussed:

Step 1: Li and He [31] established a fractional complex transform and successfully converted the FDE into ODE, so that all the analytical and semi-analytical methods associated with the advanced calculus can be implemented spontaneously. The fractional complex wave variable is:

$$u(\xi) = u(x, t), \quad \xi = \frac{k x^{\beta}}{\Gamma(1+\beta)} - \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} \tag{12}$$

where k and ω are constants. By means of eq. (12), eq. (11) can be converted to the following non-linear ODE:

$$H(u, u', u'', u''', \cdots) = 0 \tag{13}$$

where H is a polynomial in $u(\xi)$ and its derivatives, in which $u'(\xi) = \mathrm{d}u/\mathrm{d}\xi$.

Step 2: Integrate eq. (13) term by term according to possibility one or more times and put the integration constants to zero, as we search for the solitary wave solutions.

Step 3: Suppose that eq. (13) has the solution in the form:

$$u(\xi) = \sum_{k=0}^{N} a_k \left[\frac{\psi'(\xi)}{\psi(\xi)} \right]^k \tag{14}$$

where a_k $(k=0,1,2,3,\cdots)$ are arbitrary constants to be determined, such that $a_N \neq 0$, and $\psi(\xi)$ is an unknown function to be evaluated hereafter. In tanh-function method, exp-function method, (G'/G)-expansion method, fractional sub-equation method *etc.*, the solution is written subject to several previously known functions. On the other hand, in the MSE method, ψ is not previously known or not a solution of some differential equations whose solutions can be found easily. This is the distinction of this method.

Step 4: The positive integer N occurs in eq. (14) can be found by taking into account the homogeneous balance between the non-linear terms of the highest order and the highest order derivative terms come out in eq. (13).

Step 5: We compute all the non-linear terms and needed derivatives $u', u'', \cdots$, and then place them into eq. (13). This replacement yields a polynomial of $[1/\psi(\xi)]$. Setting each coefficient of this polynomial yields a system of algebraic and differential equations, which can be solved to get a_k $(k=0,1,2,3,\cdots)$ and $\psi(\xi)$.

Applications

In the present section, we investigate the closed form solutions of the fractional non-linear Cahn-Allen equation and the fractional generalized reaction Duffing equation by making use of the $[\psi'(\xi)/\psi(\xi)]$-expansion method.

The fractional non-linear Cahn-Allen equation

We consider the non-linear fractional Cahn-Allen equation [32]:

$$\mathrm{D}_t^{\alpha} u - u_{xx} + u^3 - u = 0\,, \quad t>0, \quad 0<\alpha\leq 1 \tag{15}$$

where α is a parameter related to the order of the time fractional derivative.

For the fractional Cahn-Allen equation, we introduce the following fractional complex transform:

$$u(\xi) = u(x,t)\,, \quad \xi = kx - \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} \tag{16}$$

where k and ω are constants to be determined. Using the fractional wave transformation (16), eq. (15) is transformed into the ODE:

$$-\omega u' - k^2 u'' + u^3 - u = 0 \tag{17}$$

where $u' = \mathrm{d}u/\mathrm{d}\xi$. The balance between the order of u'' and the order of u^3, yields $N=1$.

Therefore, the solution of eq. (17) takes the formal form:

$$u(\xi) = a_0 + a_1 \frac{\psi'(\xi)}{\psi(\xi)}\,, \quad a_1 \neq 0 \tag{18}$$

Inserting eq. (18) into eq. (17) yields a polynomial of $[1/\psi(\xi)]$ and setting each coefficient of this polynomial to zero provides the following system of algebraic and differential equation:

$$a_0^3 - a_0 = 0 \tag{19}$$

$$-\omega a_1 \psi'' - k^2 a_1 \psi''' + (3a_0^2 - 1) a_1 \psi' = 0 \tag{20}$$

$$(\omega + 3a_0a_1)\psi' + 3k^2\psi'' = 0 \tag{21}$$

$$-2k^2a_1(\psi')^3 + a_1^3(\psi')^3 = 0 \tag{22}$$

From eq. (19), we obtain:

$$a_0 = 0, \pm 1$$

and from eq. (22), we obtain:

$$a_1 = \pm\sqrt{2}\,k, \text{ since } \quad a_1 \neq 0$$

Therefore, for $a_0 = 0$ and $a_1 = \pm\sqrt{2}\,k$, eq. (21) yields the following value of ψ:

$$\psi(\xi) = -\frac{3k^2c_1}{\omega}\mathrm{e}^{-\frac{\omega}{3k^2}\xi} + c_2 \tag{23}$$

where c_1 and c_2 are constants of integration. Inserting the values of a_0, a_1, and ψ into (20), results the value of ω:

$$\omega = \pm\frac{3}{\sqrt{2}}k$$

Now, putting the values of a_0, a_1, ψ, and ω into solution (18), we obtain:

$$u(\xi) = \pm\sqrt{2}\,k\left(\frac{c_1\mathrm{e}^{\mp\frac{\xi}{\sqrt{2}}k}}{\mp\sqrt{2}kc_1\mathrm{e}^{\mp\frac{\xi}{\sqrt{2}}k} + c_2}\right) \tag{24}$$

Simplifying eq. (24), we obtain:

$$u(\xi) = \pm\sqrt{2}kc_1\frac{\cosh\frac{\xi}{2\sqrt{2}k} \mp \sinh\frac{\xi}{2\sqrt{2}k}}{\cosh\frac{\xi}{2\sqrt{2}k}\left(c_2 \mp \sqrt{2}kc_1\right) + \sinh\frac{\xi}{2\sqrt{2}k}\left(\pm c_2 + \sqrt{2}kc_1\right)} \tag{25}$$

Since c_1 and c_2 are integral constants, one may spontaneously choose their values. Therefore, if we choose $c_1 = \pm 1/[2(2)^{1/2}]$ and $c_2 = 3/2$, from (25), we obtain:

$$u(\xi) = \pm\frac{1}{2}\frac{\cosh\frac{\xi}{2\sqrt{2}k} \mp \sinh\frac{\xi}{2\sqrt{2}k}}{\cosh\frac{\xi}{2\sqrt{2}k} \pm 2\sinh\frac{\xi}{2\sqrt{2}k}} \tag{26}$$

On the other hand, if we put $c_1 = -1$ and $c_2 = \pm(2)^{1/2}$, from (25), we obtain:

$$u(x,t) = -\frac{1}{2}\left(1 \mp \tanh\left\{\frac{1}{2\sqrt{2}}\left[x \pm \frac{3t^\alpha}{\sqrt{2}\Gamma(1+\alpha)}\right]\right\}\right) \tag{27}$$

Again, if we set $c_1 = 1$ and $c_2 = \pm(2)^{1/2}$, from (25), we obtain:

$$u(x,t) = -\frac{1}{2}\left(1 \mp \coth\left\{\frac{1}{2\sqrt{2}}\left[x \pm \frac{3t^{\alpha}}{\sqrt{2}\Gamma(1+\alpha)}\right]\right\}\right) \tag{28}$$

For other selection of c_1 and c_2 provides many solitary wave solutions to the Cahn-Allen equation. However, for conciseness, other solutions are not recorded here.

When $a_0 = \pm 1$ and $a_1 = \pm(2)^{1/2}$, it is noteworthy to observe that the obtained solutions are identical to the solutions obtained in the above case.

The major advantage of the MSE method is that the calculations are very simple and straightforward. It does not require any symbolic computation software to facilitate the calculations as in the exp-function method, the (G'/G)-expansion, the tanh-function method, the homotopy analysis method, *etc.*

Fractional generalized reaction Duffing equation

In this subsection, we examine the closed form wave solutions by using the MSE method of the fractional generalized reaction Duffing model:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + p\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + q u(x,t) + r u^2(x,t) + s u^3(x,t) = 0, \quad t > 0, \quad 0 < \alpha \le 1 \tag{29}$$

where p, q, r, and s are constants. When $p = -1$, $q = -m^2$, $r = 0$, and $s = g^2$, eq. (29) is known as the fractional Landau-Ginzburg-Higgs equation. Whereas for $p = -1$, $q = -a$, $r = 0$, and $s = -b$, eq. (29) reduces to the classical fractional Klein-Gordon equation. Whenever $p = -1$, $q = 1$, $r = 0$, and $s = -1$, eq. (29) turns to the phi-4 equation. On the other hand, eq. (29) switches to the sine-Gordon equation, if $p = -1$, $q = 1$, $r = 0$, and $s = -1/6$. Furthermore if $p = 0$, $q = a$, $r = 0$, and $s = b$, eq. (29) changes to the Duffing equation. Since in most of the cases of real world phenomena $r = 0$, this case is studied merely in this article. Therefore, if $r = 0$, then the fractional generalized reaction Duffing model (29) becomes:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + p\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + q u(x,t) + s u^3(x,t) = 0 \tag{30}$$

In order to transform the space and time fractional PDE (30) into integer order ODE, the following wave transformation has been used:

$$u(x,t) = u(\xi), \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} \tag{31}$$

The traveling wave variable (31) transforms the eq. (30) into the subsequent ODE for $u(\xi)$:

$$(\omega^2 + pk^2)u'' + qu + su^3 = 0 \tag{32}$$

The homogeneous balance of the order u'' and u^3, yields $N = 1$. Therefore, the solution eq. (14) turns into:

$$u = a_0 + a_1\frac{\psi'}{\psi} \tag{33}$$

where a_0 and a_1 are constants such that $a_1 \ne 0$ and ψ is an underdetermined function to be computed.

Inserting eq. (33) into eq. (32), the left hand side is turned into a polynomial of $(1/\psi)^j$, $j=0,1,2,\cdots$ (for simplicity the polynomial is omitted here) and setting the coefficients of ψ^0, ψ^1, ψ^2, and ψ^3 to zero, yields:

$$q\,a_0 + s\,a_0^3 = 0 \tag{34}$$

$$(\omega^2 + pk^2)\psi''' + (q + 3a_0^2 s)\psi' = 0 \tag{35}$$

$$-(\omega^2 + pk^2)\psi'\psi'' + a_0\,a_1 s(\psi')^2 = 0 \tag{36}$$

$$2\,a_1\left(\omega^2 + p\,k^2\right)(\psi')^3 + a_1^3\,s\,(\psi')^3 = 0 \tag{37}$$

Solving eq. (34), gives:

$$a_0 = 0,\ \pm\sqrt{-\frac{q}{s}} \tag{38}$$

and the solution of eq. (37), provides:

$$a_1 = \pm\sqrt{-\frac{2(\omega^2 + pk^2)}{s}}\ ,\ \text{ since }\ a_1 \neq 0 \tag{39}$$

The value of ψ will depend on the values of a_0 and a_1, as they are connected to the system of eqs. (34)-(37).

Case 1. When $a_0 = 0$ and $a_1 = \pm[-2(\omega^2 + pk^2)/s]^{1/2}$, one can obtain ψ from eqs. (35) and (36):

$$\psi = c_1 + c_2\,\xi \tag{40}$$

where c_1 and c_2 are integral constants and $q=0$. Therefore, the solution equation (33), turns out to be:

$$u(\xi) = \pm\sqrt{-\frac{2(\omega^2 + pk^2)}{s}}\,\frac{c_2}{c_1 + c_2\xi} \tag{41}$$

Substituting the value of the fractional complex variable into eq. (41), we obtain:

$$u(x,t) = \pm c_2\sqrt{-2\frac{\omega^2 + pk^2}{s}}\left\{c_1 + c_2\left[\frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}^{-1} \tag{42}$$

The solution of eq. (42) is known as the rational function solution of the fractional generalized reaction Duffing equation.

Case 2. When $a_0 = \pm(-q/s)^{1/2}$ and $a_1 = \pm[-2(\omega^2 + pk^2)/s]^{1/2}$, eqs. (35) and (36) yield the subsequent solution of ψ in the form:

$$\psi = c_1 + c_2 \mathrm{e}^{\pm\sqrt{2q/(\omega^2 + pk^2)}\,\xi} \tag{43}$$

Thus, in this case the solution eq. (33), turns out to be:

$$u = \pm\sqrt{\frac{-q}{s}} \mp 2c_2\sqrt{\frac{-q}{s}}\,\frac{\mathrm{e}^{\sqrt{2q/(\omega^2 + pk^2)}\,\xi}}{c_1 + c_2\mathrm{e}^{\sqrt{2q/(\omega^2 + pk^2)}\,\xi}} \tag{44}$$

Simplifying solution (44), yields:

$$u=\sqrt{\frac{-q}{s}}\left[\pm 1\mp 2c_2\frac{\cosh\left(\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\,\xi\right)+\sinh\left(\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\,\xi\right)}{\cosh\left(\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\,\xi\right)(c_2+c_1)+\sinh\left(\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\,\xi\right)(c_2-c_1)}\right] \tag{45}$$

This is the general closed form solitary wave solution to the eq. (30). Since c_1 and c_2 are integral constants, one might arbitrarily choose their values and the open choices yield some distinct solutions and regenerates some other solutions existing in the literature. For instance, if we choose $c_1 = 1/3$ and $c_2 = 1/2$, from (48), we obtain:

$$u(x,t)=\pm\sqrt{\frac{-q}{s}}\frac{\cosh\left\{\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}+ +5\sinh\left\{\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}}{5\cosh\left\{\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}+ +\sinh\left\{\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}} \tag{46}$$

Again, if $c_1 = 1$ and $c_2 = 1$, then from (45), we obtain:

$$u=\pm\sqrt{\frac{-q}{s}}\tanh\left\{\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{47}$$

On the other hand, if $c_1 = -1$ and $c_2 = 1$, from (45), we obtain:

$$u=\pm\sqrt{\frac{-q}{s}}\coth\left\{\frac{1}{2}\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{48}$$

Furthermore, if $c_1 = 1$ and $c_2 = 2$, from eq. (48), we obtain:

$$u=\pm\sqrt{\frac{-q}{s}}\frac{1-2\left(\cosh\left\{\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}+ +\sinh\left\{\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}\right)}{1+2\left(\cosh\left\{\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}+ +\sinh\left\{\sqrt{\frac{2q}{\omega^2+pk^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\}\right)} \tag{49}$$

Also, if we choose different values of c_1 and c_2, we might obtain huge solutions of the fractional generalized reaction Duffing equation. In addition, if we take $p=-1$, $q=-m^2$, $r=0$, and $s=g^2$, eq. (29) is reduces to:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}-m^2u(x,t)+g^2u^3(x,t)=0, \quad t>0 \quad 0<\alpha\le 1 \tag{50}$$

called the fractional Landau-Ginzburg-Higgs equation and its closed form solutions are:

$$u(x,t)=\pm\frac{m}{g}\tanh\left\{\frac{m}{2}\sqrt{\frac{2}{k^2-\omega^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{51}$$

where $c_1=1$, $c_2=1$, and $\omega<k$. If, $c_1=-1$, $c_2=1$, we obtain:

$$u(x,t)=\pm\frac{m}{g}\coth\left\{\frac{m}{2}\sqrt{\frac{2}{k^2-\omega^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{52}$$

Furthermore if $\omega>k$, the solutions of the fractional Landau-Ginzburg-Higgs equation are:

$$u(x,t)=\pm\frac{m}{g}i\tan\left\{\frac{m}{2}\sqrt{\frac{2}{\omega^2-k^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{53}$$

$$u(x,t)=\pm\frac{m}{g}i\cot\left\{\frac{m}{2}\sqrt{\frac{2}{\omega^2-k^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{54}$$

Again, if $p=-1$, $q=-a$, $r=0$, and $s=-b$, eq. (29) transforms to:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}-a\,u(x,t)-b\,u^3(x,t)=0, \quad t>0, \quad 0<\alpha\le 1 \tag{55}$$

called the classical fractional Klein-Gordon equation and its closed form solutions are:

$$u(x,t)=\pm\sqrt{\frac{-a}{b}}\tanh\left\{\frac{1}{2}\sqrt{\frac{2a}{k^2-\omega^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{56}$$

where $c_1=1$, $c_2=1$, and $k>\omega$. On the other hand, if $c_1=-1$ and $c_2=1$, then from (45), we obtain:

$$u=\pm\sqrt{\frac{-a}{b}}\coth\left\{\frac{1}{2}\sqrt{\frac{2a}{k^2-\omega^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{57}$$

If $k<\omega$, the solutions of the classical fractional Klein-Gordon equation are:

$$u(x,t)=\pm\sqrt{\frac{a}{b}}\tan\left\{\frac{1}{2}\sqrt{\frac{2a}{\omega^2-k^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{58}$$

where $c_1=1$, $c_2=1$ and if $c_1=-1$ and $c_2=1$, we obtain:

$$u=\pm\sqrt{\frac{a}{b}}\cot\left\{\frac{1}{2}\sqrt{\frac{2a}{\omega^2-k^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{59}$$

Moreover, if $p=-1$, $q=1$, $r=0$, and $s=-1$, eq. (29) turns to:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}+u(x,t)-u^3(x,t)=0,\quad t>0,\quad 0<\alpha\le 1 \tag{60}$$

called the phi-4 equation and its closed form wave solutions are:

$$u=\pm\tanh\left\{\frac{1}{2}\sqrt{\frac{2}{\omega^2-k^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{61}$$

where $c_1=1$, $c_2=1$ and $k<\omega$. Again, when $c_1=-1$ and $c_2=1$, from (45), we attain:

$$u=\pm\coth\left\{\frac{1}{2}\sqrt{\frac{2}{\omega^2-k^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{62}$$

On the contrary, if $k<\omega$, the periodic solutions of the fractional phi-4 equation are:

$$u=\pm i\tan\left\{\frac{1}{2}\sqrt{\frac{2}{k^2-\omega^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{63}$$

where $c_1=1$, $c_2=1$. But if $c_1=-1$ and $c_2=1$, we attain:

$$u=\pm i\cot\left\{\frac{1}{2}\sqrt{\frac{2}{k^2-\omega^2}}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{64}$$

Furthermore if $p=0$, $q=a$, $r=0$, and $s=b$, eq. (29) changes to:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}+a\,u(x,t)+b\,u^3(x,t)=0,\quad t>0,\quad 0<\alpha\le 1 \tag{65}$$

called the fractional Duffing equation. The closed form wave solutions of the fractional Duffing equation are:

$$u=\pm\sqrt{\frac{-a}{b}}\tanh\left\{\frac{\sqrt{a}}{\sqrt{2}\,\omega}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{66}$$

where $c_1=1$ and $c_2=1$. When $c_1=-1$ and $c_2=1$, from (45), we achieve:

$$u=\pm\sqrt{\frac{-a}{b}}\coth\left\{\frac{\sqrt{a}}{\sqrt{2}\,\omega}\left[\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{\omega t^\alpha}{\Gamma(1+\alpha)}\right]\right\} \tag{67}$$

Conclusion

In this article, the MSE method has been successfully implemented to achieve the closed form wave solutions of the non-linear fractional Cahn-Allen equation and the fractional generalized reaction Duffing model. It has been shown that the fractional complex transformation and the MSE method are significant and important mathematical tool in investigating closed form wave solutions to a broad class of fractional-order equations. Therefore, some new exact solutions of these equations are achieved. We can underline from our understanding that the method can be implemented to other problems and is able to reduce the amount of computational work.

Acknowledgment

This work is supported by the grant Research University Grant 1001/PMATHS/8011016 and the authors acknowledge this support.

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Paper submitted: June 13, 2017
Paper revised: November 21, 2017
Paper accepted: December 5, 2017