

MOMENTS OF SAMPLE EXTREMES OF ORDER STATISTICS FROM DISCRETE UNIFORM DISTRIBUTION AND NUMERICAL RESULTS

by

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Original scientific paper
<https://doi.org/10.2298/TSCI170612291C>

In this study, the m^{th} raw moments of sample extremes of order statistics from discrete uniform distribution are obtained. The results of sample extremes of order statistics of random variable for the independent and identically discrete uniform distribution are given. Numerical values are shown in table form.

Key words: *order statistics, discrete random variables, moments, distribution function*

Introduction

The first two moments of order statistics from discrete distributions were proved by Khatri [1]. Arnold *et al.* [2] obtained the first two moments which were already obtained by Khatri [1] in a different way. All the developments on discrete order statistics lucidly accounts by Nagaraja [3, 4]. The first two moments of sample maximum of order statistics from discrete distributions were obtained by Ahsanullah and Nevzorov [5]. For n up to 15, algebraic expressions for the expected values of the sample maximum of order statistics from discrete uniform distribution were obtained by Calik and Gungor [6]. Furthermore, m^{th} raw moments of order statistics from discrete distribution were proved by Calik *et al.* [7].

The two important terms $X_{1:n}$ and $X_{n:n}$ are called extremes, and play an essential role in applications of order statistics.

Some applications of $X_{1:n}$ and $X_{n:n}$ are given:

- If the behavior of a given system is determined by the weaken link principle (any series arrangement of elements has this property), the lifetime of the system coincided with the lifetime of the weaken element in the system, *i. e.*, if X_1, X_2, \dots, X_n are the lifetimes of the components, then the lifetime of the system is given by $X_{1:n}$. Thus, the first order statistics (minimum) governs the life of the system.
- The maximum value in the sample $X_{n:n}$ is used in the study of extremes, such as rainfall, floods, sea level, and temperature.

Order statistics of discrete populations

Let X be a discrete random variable which may take the values $0, 1, 2, \dots$ with probability mass function (PMF) $f(x) = P_r(X = x)$ and the corresponding cumulative distribution function $F(x) = P_r(X \leq x)$. Suppose that X_1, X_2, \dots, X_n are n independent and identically dis-

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tributed random variable with the same PMF as that of X . Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the corresponding order statistics. The PMF $f_{r:n}(x)$ of $X_{r:n}$ can be written:

$$f_{r:n}(x) = P\{X_{r:n} = x\} = F_{r:n}(x) - F_{r:n}(x-1) \quad (1)$$

$$\sum_{i=r}^n \binom{n}{i} [F(x)]^i \{[1-F(x)]^{n-i} - [F(x-1)]^i [1-F(x-1)]^{n-i}\} \quad (2)$$

Using the identity of eq. (2) can be given:

$$\sum_{i=r}^n \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i} = \int_0^{F(x)} r \binom{n}{i} u^{r-1} (1-u)^{n-r} du, \quad 0 < u < 1 \quad (3)$$

Thus, we get:

$$f_{r:n}(x) = C(r, n) \int_{F(x-1)}^{F(x)} u^{r-1} (1-u)^{n-r} du \quad (4)$$

where $C(r, n) = n! / (r-1)!(n-r)!$.

The smallest and the largest order statistics in samples of size n are of special interest in numerous practical applications. When $r = 1$ and $r = n$, the PMF of 1th and n th order statistics from eq. (4) are obtained, respectively:

$$f_{1:n}(x) = n \int_{F(x-1)}^{F(x)} (1-u)^{n-1} du = [\bar{F}(x-1)]^n - [\bar{F}(x)]^n, \quad \bar{F}(x) = 1 - F(x) \quad (5)$$

and

$$f_{n:n}(x) = n \int_{F(x-1)}^{F(x)} u^{n-1} du = [F(x)]^n - [F(x-1)]^n \quad (6)$$

Theorems for the moments of order statistics

Suppose X_1, X_2, \dots, X_n are independent and identically distributed discrete uniform random variables with $f(x) = 1/k$ and $F(x) = x/k$, $x = 1, \dots, k$. Thus, we can write eq. (7).

We have:

$$f_{r:n}(x) = \int_{F(x-1)}^{F(x)} C(r, n) u^{r-1} (1-u)^{n-r} du = \int_{(x-1)/k}^{x/k} C(r, n) u^{r-1} (1-u)^{n-r} du \quad (7)$$

For $r = 1$ and $r = n$, eqs. (8) and (9) can be written, respectively, [8, 9]:

$$f_{1:n}(x) = \left(\frac{k+1-x}{k} \right)^n - \left(\frac{k-x}{k} \right)^n, \quad x = 1, 2, \dots, k \quad (8)$$

$$f_{n:n}(x) = \left(\frac{x}{k} \right)^n - \left(\frac{x-1}{k} \right)^n, \quad x = 1, 2, \dots, k \quad (9)$$

Here, expressions that concerning the moments of $X_{1:n}$ and $X_{n:n}$ are demonstrated. Let us denote the single moment $E(X_{r:n}^m)$ by $\mu_{r:n}^{(m)}$ ($1 \leq r \leq n$, $m \geq 1$). Then, from the probability mass function of $X_{r:n}$ in eq. (4), we have:

$$\mu_{r:n}^{(m)} = E(X_{r:n}^m) = \sum_{x=0}^{\infty} x^m f_{r:n}(x) \quad (10)$$

For convenience, $\mu_{r:n}$ for $\mu_{r:n}^{(1)}$ and $\sigma_{r:n}^2$ for variance of $X_{r:n}$ will also be used.

Theorem 1. Let X_1, X_2, \dots, X_n be random variables from a discrete uniform distributions and $X_{1:n}$ be the sample minimum of order statistics corresponding to these random variables. Then, m^{th} moment of $X_{1:n}$:

$$\mu_{1:n}^{(m)} = E(X_{1:n}^m) = \sum_{i=1}^k [i^m - (i-1)^m] \left(\frac{k+1-i}{k} \right) \quad (11)$$

whenever the moment on the left-hand side is assumed to exist.

Proof. If expression in eq. (8) of $f_{1:n}(x)$ is written in definition of m^{th} moment of $X_{1:n}$:

$$\mu_{1:n}^{(m)} = E(X_{1:n}^m) = \sum_{i=1}^k x^m \left[\left(\frac{k+1-x}{k} \right)^n - \left(\frac{k-x}{k} \right)^n \right] \quad (12)$$

If the sum on the right-hand side is opened and simplification is made:

$$\begin{aligned} \mu_{1:n}^{(m)} &= E(X_{1:n}^m) = \\ &= 1^m \left[\left(\frac{k}{k} \right)^n - \left(\frac{k-1}{k} \right)^n \right] + 2^m \left[\left(\frac{k-1}{k} \right)^n - \left(\frac{k-2}{k} \right)^n \right] + \dots + k^m \left[\left(\frac{1}{k} \right)^n \right] = \\ &= 1^m \left(\frac{k}{k} \right)^n + [2^m - 1^m] \left(\frac{k-1}{k} \right)^n + \dots + [k^m - (k-1)^m] \left(\frac{1}{k} \right)^n = \\ &= \sum_{i=1}^k [i^m - (i-1)^m] \left(\frac{k+1-i}{k} \right)^n \end{aligned} \quad (13)$$

Theorem 2. Let X_1, X_2, \dots, X_n be random variables from a discrete uniform distributions and $X_{n:n}$ be the sample maximum of order statistics corresponding to these random variables. Then, m^{th} moment of $X_{n:n}$:

$$\mu_{n:n}^{(m)} = E(X_{n:n}^m) = \sum_{x=1}^{k-1} [i^m - (i+1)^m] \left(\frac{i}{k} \right)^n + k^m \quad (14)$$

whenever the moment on the left-hand side is assumed to exist.

Proof. If expression in eq. (9) of $f_{n:n}(x)$ is written in definition of m^{th} moment of $X_{n:n}$:

$$\mu_{n:n}^{(m)} = E(X_{n:n}^m) = \sum_{x=1}^k x^m \left[\left(\frac{x}{k} \right)^n - \left(\frac{x-1}{k} \right)^n \right] \quad (15)$$

If the sum on the right-hand side is opened and simplification is made:

$$\begin{aligned} \mu_{n:n}^{(m)} &= E(X_{n:n}^m) = \\ &= 1^m \left[\left(\frac{1}{k} \right)^n \right] + 2^m \left[\left(\frac{2}{k} \right)^n - \left(\frac{1}{k} \right)^n \right] + \dots + k^m \left[\left(\frac{k}{k} \right)^n - \left(\frac{k-1}{k} \right)^n \right] = \end{aligned}$$

$$\begin{aligned}
&= [1^m - 2^m] \left(\frac{1}{k}\right)^n + [2^m - 3^m] \left(\frac{2}{k}\right)^n = \\
&= \sum_{i=1}^k [i^m - (i-1)^m] \left(\frac{k+1-i}{k}\right)^n + \dots + [(k-1)^m - k^m] \left(\frac{k-1}{k}\right)^n + k^m = \\
&= \sum_{i=1}^k [i^m - (i+1)^m] \left(\frac{i}{k}\right)^n + k^m \tag{16}
\end{aligned}$$

Results for moments of sample extremes

In this section, results concerning to moments of $X_{1:n}$ and $X_{n:n}$ are given, respectively.

Result 1.

$$\mu_{1:n} = E(X_{1:n}) = \sum_{i=1}^k \left(\frac{k+1-i}{k}\right)^n \tag{17}$$

$$\mu_{1:n}^{(2)} = E(X_{1:n}^2) = \sum_{i=1}^k (2i-1) \left(\frac{k+1-i}{k}\right)^n \tag{18}$$

Proof. In the *Theorem 1*, if $m=1$ and $m=2$ are taken, respectively, eqs. (17) and (18) are obtained.

Result 2.

$$\mu_{n:n} = E(X_{n:n}) = \sum_{i=1}^{k-1} (-1) \left(\frac{i}{k}\right)^n + k \tag{19}$$

$$\mu_{n:n}^{(2)} = E(X_{n:n}^2) = \sum_{i=1}^{k-1} (-2i-1) \left(\frac{i}{k}\right)^n + k^2 \tag{20}$$

Proof. In the *Theorem 2*, if $m=1$ and $m=2$ are taken, respectively, eqs. (19) and (20) are obtained.

According to this, given results for expected values and variances of sample minimum and sample maximum of order statistics from discrete uniform distribution are shown in the tab. 1.

Table 1. Expected values and variances of sample extremes

k	n	$\mu_{1:n}$	$\sigma_{1:n}^2$	$\mu_{n:n}$	$\sigma_{n:n}^2$
10	1	5.5000	8.2500	5.5000	8.2500
	3	3.0250	3.7084	7.9750	3.7084
	5	2.2083	1.9288	8.7918	1.9288
	10	1.4914	0.6229	9.5086	0.6229
20	1	10.500	33.250	10.500	33.250
	3	5.5125	14.958	15.487	14.958
	5	3.8542	7.8810	17.145	7.8810
	10	2.3597	2.6872	18.640	2.6872
100	1	50.500	833.25	50.500	833.25
	3	25.502	374.95	75.497	374.95
	5	17.170	198.35	83.829	198.35
	10	9.5992	68.802	91.400	68.802

Conclusion

The current study presents the obtained m^{th} raw moments of extremes of order statistics from discrete uniform distributions. From the statistical point of view, the moments of order statistics carry great importance to estimate means and variances for the discrete uniform distributions. For the purpose of giving an example for discrete uniform distributions, we showed the results in this study.

In conclusion, all the moments for sample extremes of order statistics from the discrete uniform distribution can be achieved. Further studies may focus on a software program computing the means and variance of order statistics from any discrete uniform distribution.

Nomenclature

$f(x)$ – probability mass function
 $F(x)$ – distribution function

Greek symbols

$\mu_{r:n}$ – expected value of r^{th} order statistics
 $\mu_{r:n}^m$ – m^{th} moment of r^{th} order statistics
 $\sigma_{r:n}$ – variance of r^{th} order statistics

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