NUMERICAL INVESTIGATION OF THE INVERSE NODAL PROBLEM BY CHEBYSHEV INTERPOLATION METHOD

by

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In this study, we deal with the inverse nodal problem for Sturm-Liouville equation with eigenparameter-dependent and jump conditions. Firstly, we obtain reconstruction formulas for potential function, q, under a condition and boundary data, α, as a limit by using nodal points to apply the Chebyshev interpolation method. Then, we prove the stability of this problem. Finally, we calculate approximate solutions of the inverse nodal problem by considering the Chebyshev interpolation method. We then present some numerical examples using Matlab software program to compare the results obtained by the classical approach and by Chebyshev polynomials for the solutions of the problem.

Key words: *inverse nodal problem, stability, Sturm-Liouville equation,*
Chebyshev interpolation method, approximate solutions

Introduction

Inverse problem is a research area dealing with inversion of models or data. An inverse problem is a mathematical framework that is used to obtain information about a physical object or system from observed measurements. The solution to this problem is useful because it generally provides information about a physical parameter that we can not directly observe. Thus, inverse problems are some of the most important and well studied mathematical problems in science and mathematics. There are many different applications, including but not limited to, medical imaging, geophysics, computer vision, astronomy, and non-destructive testing. We have two approaches related to inverse problems, one is to study inverse eigenvalue problem [1-3], and the other one is to study inverse nodal problem. Inverse nodal problem was first solved by McLaughlin [4] for the Sturm-Liouville (SL) equation. She proved that the potential function of the SL problem could obtained by a given dense subset of nodes for the eigenfunction up to a constant. Recently, many authors have studied the inverse nodal problem for several operators [5-10]. In [11, 12], the authors studied inverse problems for some differential operators with the eigenparameter-dependent jump conditions.

Let us consider the boundary value problem $l = l(q, H_0, H_1, H_2, \alpha)$ generated by the SL equation:

$$ly(x,\lambda) = -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0,1) \tag{1}$$

with the boundary conditions:

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$$U(y) := y(0) = 0 \tag{2}$$

$$V(y) := \lambda[y'(1) + H_0 y(1)] - H_1 y'(1) - H_2 y(1) = 0 \tag{3}$$

and the jump conditions:

$$\begin{cases} y\left(\dfrac{1}{2}+0\right) = \alpha y\left(\dfrac{1}{2}-0\right) \\ y'\left(\dfrac{1}{2}+0\right) = \alpha^{-1} y'\left(\dfrac{1}{2}-0\right) - (\lambda+\beta) y\left(\dfrac{1}{2}-0\right) \end{cases} \tag{4}$$

where $q(x)$ is a real valued bounded function in $L_2(0,1)$, α, β, and H_i, $i = 0,1,2$ are real numbers, $\alpha > 0$, $H_0 H_1 - H_2 > 0$, and λ is a spectral parameter. Ozkan and Keskin [12] considered the boundary value problem, eqs. (1)-(4), and they solved an inverse nodal problem to reconstruct some of the coefficients. In this study, we will consider the boundary value problem (1)-(4) to give asymptotic formulas for nodal lengths and reconstruction formulas for potential function, q, and boundary data, α, from a different perspective. Then, we will examine the Lipschitz stability of this given boundary value problem.

In recent years, many authors studied inverse SL problems and solved these problems by using numerical methods such as Numerov's method, Newton type method, a finite difference method, Rayleigh-Ritz method and other methods [13-23]. In this study, we obtain the numerical values of the potential function, $q(x)$, under the previous conditions by applying the Chebyshev interpolation method. The Chebyshev interpolation method was used in [24] for calculating the solution of the integro-differential equations. We apply this method to obtain the solution of the inverse nodal problem for SL equation with jump conditions.

Theorem 1. [12] Suppose that $\varphi(x,\lambda)$ is the solution of eq. (1) under the initial conditions:

$$\varphi(0,\lambda) = 0, \quad \varphi'(0,\lambda) = 1 \tag{5}$$

and jump conditions given by eq. (4). Then, the following asymptotic relations are satisfied for $\varphi(x,\lambda)$ as $|\lambda| \to \infty$, when $x < 1/2$

$$\varphi(x,\lambda) = \frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}} - \frac{\cos\left(\sqrt{\lambda}x\right)}{2\lambda}\int_0^x q(t)\mathrm{d}t + o\left(\frac{1}{\lambda}\mathrm{e}^{\tau x}\right) \tag{6}$$

and in case of $x > 1/2$:

$$\varphi(x,\lambda) = \frac{1}{2}\left\{\cos\left(\sqrt{\lambda}x\right) - \cos\left[\sqrt{\lambda}(1-x)\right]\right\} + I_1(x)\frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}} +$$
$$+ I_2(x)\frac{\sin\left[\sqrt{\lambda}(1-x)\right]}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\mathrm{e}^{\tau x}\right) \tag{7}$$

where $\tau = \left|\operatorname{Im}\sqrt{\lambda}\right|$, $\alpha^{\pm} = (1/2)[\alpha \pm (1/\alpha)]$,

$$I_1(x) = \alpha^{+} + \frac{1}{4}\int_0^x q(t)\mathrm{d}t \quad \text{and} \quad I_2(x) = \alpha^{-} - \frac{1}{2}\int_0^{1/2} q(t)\mathrm{d}t + \frac{1}{4}\int_0^x q(t)\mathrm{d}t$$

Theorem 2. [12] Let $\{\lambda_n\}_{n\geq 0}$ be the set of eigenvalues of the problem (1)-(4). The sequence $\{\lambda_n\}_{n\geq 0}$ satisfies the following asymptotic estimates as $n \to \infty$:

$$\sqrt{\lambda_n} = (n-1)\pi + \frac{A}{(n-1)\pi} + o\left(\frac{1}{n}\right) \tag{8}$$

$$\frac{1}{\sqrt{\lambda_n}} = \frac{1}{(n-1)\pi} - \frac{A}{(n-1)^2\pi^2}\frac{1}{(n-1)\pi} + o\left(\frac{1}{n^3}\right)$$

where $A = w_1 + (-1)^{n-1} w_2$, $w_1 = 2I_1(1) + H_0$, and $w_2 = 2I_2(1) + H_0$.

Let $X = \{x_n^j : n = 2m,\ m \in \mathbb{N}\}$ be the set of nodal points with even indexes of the eigenfunction and X_0 be a dense subset of X in $(0,1)$. Ozkan and Keskin [12] gave an asymptotic formula for the elements of X_0 as follows.

Lemma 3. [12] The elements of X_0 satisfy the below asymptotic expansion for a sufficiently large n and $x_n^j \in (0,1/2)$:

$$x_n^j = \frac{j}{n-1} - \frac{w_1 - w_2}{(n-1)^2\pi^2}\frac{j}{n-1} + \frac{1}{2(n-1)^2\pi^2}\int_0^{x_n^j} q(t)\mathrm{d}t + o\left(\frac{1}{n^2}\right) \tag{9}$$

and for $x_n^j \in (1/2,1)$:

$$x_n^j = \frac{j+\frac{1}{2}}{n-1} - \frac{w_1 - w_2}{(n-1)^2\pi^2}\frac{j+\frac{1}{2}}{n-1} + \frac{w_1 - w_2 + 2I_1(x_n^j) + 2I_2(x_n^j)}{2(n-1)^2\pi^2} + o\left(\frac{1}{n^2}\right) \tag{10}$$

Some asymptotic expansions for nodal lengths and coefficients

In this section, we will try to obtain some asymptotic results for nodal lengths and some reconstruction formulas for the potential function, q, and boundary data, α. Assume that $L = \{l_n^j : n = 2m,\ m \in \mathbb{N}\}$ is the set of nodal lengths with even indexes of the eigenfunction. We define nodal length by $l_n^j = x_n^{j+1} - x_n^j$ and $j = j_n(x) = \max\left\{k : x_n^k \le x\right\}$.

Lemma 1. Associated nodal lengths of the given operator have the following asymptotic expansion for a sufficiently large *n* when $x_n^j \in (0,1/2)$:

$$l_n^j = \frac{1}{n-1} - \frac{w_1 - w_2}{(n-1)^3\pi^2} + \frac{1}{2(n-1)^2\pi^2}\int_{x_n^j}^{x_n^j+1} q(t)\mathrm{d}t + o\left(\frac{1}{n^2}\right)$$

and for $x_n^j \in (1/2,1)$:

$$l_n^j = \frac{1}{n-1} - \frac{w_1 - w_2}{(n-1)^3\pi^2} + \frac{I_1(x_n^{j+1}) - I_1(x_n^j) + I_2(x_n^{j+1}) - I_2(x_n^j)}{(n-1)^2\pi^2} + o\left(\frac{1}{n^2}\right)$$

Proof. If we use the definition of nodal length and eq. (9), the proof can be made easily. Furthermore, it can be shown that l_n^j has the following asymptotic formula:

$$l_n^j = \frac{1}{n-1} - \frac{w_1 - w_2}{(n-1)^3\pi^2} + \frac{I_1(x_n^{j+1}) - I_1(x_n^j) + I_2(x_n^{j+1}) - I_2(x_n^j)}{(n-1)^2\pi^2} + o\left(\frac{1}{n^2}\right) \tag{11}$$

for $x_n^j \in (1/2,1)$. This completes the proof.

Corollary 2. Notice that if we use the definitions of I_1 and I_2 we get:

$$I_1(x_n^{j+1}) - I_1(x_n^j) = \frac{1}{4}\int_{x_n^j}^{x_n^j+1} q(t)\mathrm{d}t = I_2(x_n^{j+1}) - I_2(x_n^j)$$

So, we see that asymptotic expansions of nodal lengths are the same for $x_n^j \in (0,1/2)$ and $x_n^j \in (1/2,1)$.

Theorem 3. Let $X = \{x_n^j\}$ be the set of nodes with even indexes of the eigenfunction for the boundary value problem, eqs. (1)-(4). Then, the boundary data α can be reconstructed:

$$\alpha^- - \alpha^+ = \lim_{\frac{j}{n-1}\to\frac{1}{2}} (n-1)^2 \pi^2 \left(x_n^j - \frac{j}{n-1} \right)$$

for $x_n^j \in (0,1/2)$ and

$$\alpha^- - \alpha^+ = \lim_{\frac{j+\frac{1}{2}}{n-1}\to\frac{1}{2}} (n-1)^2 \pi^2 \left(x_n^j - \frac{j+\frac{1}{2}}{n-1} \right)$$

for $x_n^j \in (1/2,1)$.

Proof. Let $x_n^j \in (0,1/2)$. Using eq. (9), we have:

$$(n-1)^2 \pi^2 \left(x_n^j - \frac{j}{n-1} \right) = -\left(w_1 - w_2 \right) \frac{j}{(n-1)} + \frac{1}{2} \int_0^{x_n^j} q(t)\mathrm{d}t + o(1)$$

By using definitions of w_1 and w_2, we get:

$$(n-1)^2 \pi^2 \left(x_n^j - \frac{j}{n-1} \right) = -\left[2\alpha^+ - 2\alpha^- + \int_0^{1/2} q(t)\mathrm{d}t \right] \frac{j}{n-1} + \frac{1}{2} \int_0^{x_n^j} q(t)\mathrm{d}t + o(1)$$

Hence, letting $\left[j/(n-1) \right] \to 1/2$, it:

$$\alpha^- - \alpha^+ = \lim_{\frac{j}{n-1}\to\frac{1}{2}} (n-1)^2 \pi^2 \left(x_n^j - \frac{j}{n-1} \right)$$

Similarly, we can easily obtain the reconstruction formulas of the boundary data α for $x_n^j \in (1/2,1)$.

Lemma 4. [25] Suppose that $q \in L_1(0,1)$. Then, for almost every $x \in (0,1)$ with $j = j_n(x)$:

$$\lim_{n\to\infty} \frac{\sqrt{\lambda_n}}{\pi} \int_{x_n^j}^{x_n^j+1} q(t)\mathrm{d}t = q(x)$$

Theorem 5. The potential function $q \in L_1(0,1)$ satisfies:

$$q(x) = \lim_{n\to\infty} 2\left\{ (n-1)^2 \pi^2 [(n-1)l_n^j - 1] + w_1 - w_2 \right\}$$

for almost every $x \in (0,1)$, with $j = j_n(x)$.

Proof. We know that the given boundary value problem has the same nodal length for $x_n^j \in (0,1/2)$ and $x_n^j \in (1/2,1)$. By using *Lemma 1*, we obtain:

$$(n-1)l_n^j - 1 = -\frac{w_1 - w_2}{(n-1)^2 \pi^2} + \frac{1}{2(n-1)\pi^2} \int_{x_n^j}^{x_n^j+1} q(t)\mathrm{d}t + o\left(\frac{1}{n^2} \right)$$

so that

$$(n-1)^2 \pi^2 [(n-1)l_n^j - 1] = -\left(w_1 - w_2 \right) + \frac{(n-1)\pi}{2\pi} \int_{x_n^j}^{x_n^j+1} q(t)\mathrm{d}t + o(1)$$

Using *Lemma 4*, we have:

$$q(x)=\lim_{n\to\infty} 2\left\{(n-1)^2\pi^2[(n-1)l_n^j-1]+w_1-w_2\right\}$$

for almost every $x\in(0,1)$. This completes the proof.

Theorem 6. Suppose that $X=\{x_n^j\}$ is the set of nodes with even indexes for problem of eqs. (1)-(4) with $q\in L_1(0,1)$. Let us define:

$$F_n(x)=2(n-1)^2\pi^2[(n-1)l_n^j-1]+8\alpha^{+}-8\alpha^{-}+4\int_0^{1/2} q(t)\mathrm{d}t$$

Then, $\{F_n\}$ converges to q pointwise almost everywhere and in $L_1(0,1)$.

Proof. By using the asymptotic equation for eigenvalues, we have:

$$2\left[\lambda_n\left(\frac{\sqrt{\lambda_n}l_n^j}{\pi}-1\right)+w_1-w_2\right]=2\lambda_n\left\{\frac{l_n^j}{\pi}\left[(n-1)\pi+\frac{A}{(n-1)\pi}+o\left(\frac{1}{n}\right)\right]-1\right\}+2w_1-2w_2=$$

$$=2\lambda_n[(n-1)l_n^j-1]+2A(n-1)l_n^j+2w_1-2w_2+o(1)$$

Finally, if we use the definitions of $A, w_1, \text{and}\, w_2$, this implies:

$$2(n-1)^2\pi^2[(n-1)l_n^j-1]\to q(x)-8\alpha^{+}+8\alpha^{-}-4\int_0^{1/2} q(t)\mathrm{d}t$$

pointwise almost everywhere and in $L_1(0,1)$.

Main results regarding Lipschitz stability of the inverse nodal problem

In this section, we consider Lipschitz stability of the inverse nodal problem for SL eq. (1). Lipschitz stability is about a continuity between two metric spaces. Therefore, we have to construct two metric spaces first. To show this continuity, we shall use a homeomorphism between these two spaces. Stability problems have been studied by many researches [25-31]. With respect to the question of stability we have to decide whether the solution depends on the data continuously. Stability is necessary if we want to be sure that a variation of the given data in a sufficiently small range leads to an arbitrarily small change in the solution [32]. That is why, we want to deal with the stability issue of eqs. (1)-(4).

Definition 1. [25] Let $N'=N-\{1\}$. We denote:

- $\Omega=\{q\in L_1(0,1):q(x)\text{ is the potential function of the eq. (1)}\}$, Σ is the collection of all double sequences defined:

  $$X=\{X_n^k:k=1,2,...,n:n\in N'\}$$

  such that $n=2m, m\in\mathbb{N}$ and $0<X_n^1<X_n^2<...<X_n^{n-1}<1$, for each n.
- Let $X\in\Sigma$ and define $X=\{X_n^k\}$, $L=\{L_n^k\}$ where $L_n^k=X_n^{k+1}-X_n^k$ and $I_n^k=\left(X_n^k,X_n^{k+1}\right)$. We say X is quasinodal to some $q\in\Omega$ if X is an admissible sequence of nodes and satisfies:

  X and L have the following asymptotic uniformly for k as $n\to\infty$

  $$X_n^k=\frac{k}{n-1}+O\left(\frac{1}{n^2}\right),\ L_n^k=\frac{1}{n-1}+O\left(\frac{1}{n^2}\right),\ k=1,2,...,n$$

when $X_n^k \in (0,1/2)$ and

$$X_n^k = \frac{k+\frac{1}{2}}{n-1} + O\left(\frac{1}{n^2}\right),\ L_n^k = \frac{1}{n-1} + O\left(\frac{1}{n^2}\right),\ k = 1,2,...,n$$

when $X_n^k \in (1/2,1)$ of the eqs. (1)-(4).

We then we use L_n^k (the grid lengths) to go with X_n^k in the same way I_n^k (the nodal lengths) goes with x_n^k. We shall see that they have highly similar properties.

Let us define:

$$\Lambda \equiv \{X \in \Sigma : X \text{ is quasinodal to some } q \in \Omega\}.$$

We denote the space, Ω, as a collection of all operators, l, and the space, Λ, as a collection of all admissible double sequences of nodes such that corresponding functions, F_n, are convergent in L_1. We shall then introduce some metrics on Ω and Λ. First, define:

$$d_\Omega(H,\bar{H}) = \|q-\bar{q}\|_1 + |\alpha^+ - \bar{\alpha}^+| + |\alpha^- - \bar{\alpha}^-| + |H_0 - \bar{H}_0|$$

for any $H = (q,\alpha^+,\alpha^-,H_0)$ and $\bar{H} = (\bar{q},\bar{\alpha}^+,\bar{\alpha}^-,\bar{H}_0)$. Obviously, d_Ω is a metric on Ω. A pseudometric d_Λ on Λ will then be defined. Essentially, $d_\Lambda(X,\bar{X})$ is so close to:

$$d_0(X,\bar{X}) = \lim_{n\to\infty} 2(n-1)^2 \pi^2 \sum_{k=1}^{n-1} |L_n^k - \bar{L}_n^k|$$

where $L_n^k = X_n^{k+1} - X_n^k$, $\bar{L}_n^k = \bar{X}_n^{k+1} - \bar{X}_n^k$.

If we define $X \sim \bar{X}$, if and only if $d_\Lambda(X,\bar{X}) = 0$, then $\sim$ is an equivalence relation on Λ and d_Λ would be a metric for the partition set $\Lambda/\sim$. Let ϕ be a homeomorphism that maps Ω onto $\Lambda/\sim$ ϕ is called nodal map.

Lemma 2. Let $X \sim \bar{X}$ belong to the same cases.

- $\chi_{n,k} = |X_n^k - \bar{X}_n^k| = O(1/n^2)$ as $n \to \infty$.
- For all $x \in (0,1)$,

$$|J_n(x) - \bar{J}_n(x)| \le 1$$

for sufficiently large n.

Proof. The proof can be obtained easily by using similar approach as in [25].

Lemma 3. [25] Let $X,\bar{X} \in \Lambda$. Then, d_Λ is a pseudometric on Λ.

Definition 4. Suppose that $X,\bar{X} \in \Lambda$ with L_n^k and $\bar{L}_n^k$ as their respective grid lengths. Let:

$$S_n\left(X,\bar{X}\right) = 2(n-1)^2 \pi^2 \sum_{k=1}^{n-1} |L_n^k - \bar{L}_n^k|$$

Define:

$$d_0(X,\bar{X}) = \overline{\lim_{n\to\infty}}\ S_n\left(X,\bar{X}\right) \text{ and } d_\Lambda(X,\bar{X}) = \overline{\lim_{n\to\infty}} \frac{S_n\left(X,\bar{X}\right)}{1+S_n\left(X,\bar{X}\right)}$$

We obtain this metric by evaluating $\|q-\bar{q}\|_1$ in *Theorem 5*.This definition was first made by [25, 28]. After the forward theorems, we can say that the inverse nodal problem for operator (1) is stable.

Theorem 5. If $X,\bar{X}\in\Lambda$ are asymptotically nodal to $(q,\alpha^{+},\alpha^{-},H_0)$ and $(\bar{q},\bar{\alpha}^{+},\bar{\alpha}^{-},\bar{H}_0)$, respectively, then:

$$\|q-\bar{q}\|_1+4\left|\alpha^{+}-\bar{\alpha}^{+}\right|+4\left|\alpha^{-}-\bar{\alpha}^{-}\right|+4\left|H_0-\bar{H}_0\right|+d_0(X,\bar{X})+2\|q-\bar{q}\|_{1/2}=0$$

Proof. We only need to consider when $X,\bar{X}\in\Lambda$ belong to the same case. Without loss of generality, let $X,\bar{X}$ belong to case $X_n^k\in(0,1/2)$. For almost every x in (0,1), we can write:

$$q(x)-\bar{q}(x)=\lim_{n\to\infty}\left\{2(n-1)^3\pi^2[L_n^{J_n(x)}-\bar{L}_n^{\bar{J}_n(x)}]+2\left(w_1-\bar{w}_1\right)+2\left(w_2-\bar{w}_2\right)\right\}$$

Hence, by Fatou's *Lemma*:

$$\|q(x)-\bar{q}(x)\|=\int_0^1\left|q(x)-\bar{q}(x)\right|\mathrm{d}x\le$$

$$\le\varliminf_{n\to\infty}\left\{2(n-1)^3\pi^2\int_0^1\left|L_n^{J_n(x)}-\bar{L}_n^{\bar{J}_n(x)}\right|\mathrm{d}x+2\int_0^1\left|w_1-\bar{w}_1\right|\mathrm{d}x+2\int_0^1\left|w_2-\bar{w}_2\right|\mathrm{d}x\right\}\le$$

$$\le\varlimsup_{n\to\infty}\left\{2(n-1)^3\pi^2\int_0^1\left|L_n^{J_n(x)}-\bar{L}_n^{J_n(x)}\right|\mathrm{d}x+2(n-1)^3\pi^2\int_0^1\left|\bar{L}_n^{J_n(x)}-\bar{L}_n^{\bar{J}_n(x)}\right|\mathrm{d}x\right\}+$$

$$+\varlimsup_{n\to\infty}2\left\{\int_0^1\left|w_1-\bar{w}_1\right|\mathrm{d}x+\int_0^1\left|w_2-\bar{w}_2\right|\mathrm{d}x\right\}\tag{12}$$

On the right side of the inequality, the first and second terms can be written:

$$\int_0^1\left|L_n^{J_n(x)}-\bar{L}_n^{J_n(x)}\right|\mathrm{d}x=2(n-1)^2\pi^2\sum_{k=1}^{n-1}\left|L_n^k-\bar{L}_n^k\right|+o\left(\frac{1}{n^2}\right)\tag{13}$$

$$\int_0^1\left|\bar{L}_n^{J_n(x)}-\bar{L}_n^{\bar{J}_n(x)}\right|\mathrm{d}x=o(1)\tag{14}$$

Moreover, by using the definitions of w_1 and w_2, we get:

$$\int_0^1\left|w_1-\bar{w}_1\right|\mathrm{d}x\le2\left|\alpha^{+}-\bar{\alpha}^{+}\right|+\left|H_0-\bar{H}_0\right|+\tfrac{1}{2}\|q-\bar{q}\|_1\tag{15}$$

$$\int_0^1\left|w_2-\bar{w}_2\right|\mathrm{d}x\le2\left|\alpha^{-}-\bar{\alpha}^{-}\right|+\left|H_0-\bar{H}_0\right|+\tfrac{1}{2}\|q-\bar{q}\|_1+\|q-\bar{q}\|_{1/2}\tag{16}$$

If we consider last equalities in eq. (12), we get:

$$\|q-\bar{q}\|_1\le2\|q-\bar{q}\|_1+4\left|\alpha^{+}-\bar{\alpha}^{+}\right|+4\left|\alpha^{-}-\bar{\alpha}^{-}\right|+4\left|H_0-\bar{H}_0\right|+d_0(X,\bar{X})+2\|q-\bar{q}\|_{1/2}+$$

$$+\lim_{n\to\infty}\left[2(n-1)^2\pi^2\sum_{k=1}^{n-1}\left|L_n^k-\bar{L}_n^k\right|+o\left(\frac{1}{n^2}\right)\right]$$

and finally, we have:

$$0\le\|q-\bar{q}\|_1+4\left|\alpha^{+}-\bar{\alpha}^{+}\right|+4\left|\alpha^{-}-\bar{\alpha}^{-}\right|+4\left|H_0-\bar{H}_0\right|+d_0(X,\bar{X})+2\|q-\bar{q}\|_{1/2}\tag{17}$$

This completes the proof.

Theorem 6. The metric spaces (Ω, d_Ω) and $\left(\Lambda/\sim,\ d_\Lambda\right)$ are homeomorphic to each other.

Proof. It is clear from *Theorem 5* that $d_\Lambda(X,\overline{X})=0$ if and only if $H=\overline{H}$. Thus, the partition set $\Lambda/\sim$ is in one to one correspondence with Ω. On the other hand, if $d_\Omega(H,\overline{H})$ is so small, then $H,\overline{H}$ also belong to the same case. Hence by *Theorem 5*, and by the definition of $d_0, d_\Lambda(X,\overline{X})$ is also small. This completes the proof. Consequently, the inverse nodal problem for SL equation with the eigenparameter-dependent jump conditions is stable.

Numerical algorithm

In this section, we describe a numerical method based on the Chebyshev interpolation method for solving the inverse SL problem (1)-(4) with energy dependent jump conditions by using a dense subset of the nodal points. We consider the following inverse nodal problem.

Inverse problem. Given the nodal points $\{x_n^j\},\ j=1,2,...,n-1,\ n=2m,\ m\in\mathbb{N}$ construct the potential function $q(x)$.

Since the nodal points $\{x_n^j\},\ j=1,2,...,n-1$ are the zeroes of the n^{th} eigenfunction $\varphi(x,\lambda_n)$, then, we can write:

$$\begin{cases}\varphi(x_n^j,\lambda_n)=0,\ x_n^j\in(0,1/2),\ n=2m\\ \varphi(x_n^j,\lambda_n)=0,\ x_n^j\in(1/2,1),\ n=2m\end{cases}$$

Thus, using eqs. (6) and (7), we get:

$$\begin{cases}\displaystyle\int_0^{x_n^j} q(t)\cos\left(\sqrt{\lambda_n}x_n^j\right)\mathrm{d}t\cong 2\sqrt{\lambda_n}\sin\left(\sqrt{\lambda_n}x_n^j\right), \qquad x_n^j\in(0,1/2),\\ \displaystyle\int_0^{x_n^j} q(t)\left\{\sin\left(\sqrt{\lambda_n}x_n^j\right)+\sin\left[\sqrt{\lambda_n}(1-x_n^j)\right]\right\}\mathrm{d}t-2\int_0^{1/2} q(t)\sin\left[\sqrt{\lambda_n}(1-x_n^j)\right]\mathrm{d}t\cong\\ \cong -2\sqrt{\lambda_n}\left\{\cos\left(\sqrt{\lambda_n}x_n^j\right)-\cos\left[\sqrt{\lambda_n}\left(1-x_n^j\right)\right]\right\}-4\alpha^+\sin\left(\sqrt{\lambda_n}x_n^j\right)-\\ -4\alpha^-\sin\left[\sqrt{\lambda_n}\left(1-x_n^j\right)\right]\mathrm{d}t, \qquad x_n^j\in(1/2,1)\end{cases} \tag{18}$$

In the previous integral equation, the potential function, q, is an unknown function. In order to obtain the solution of the inverse nodal problem, it is sufficient to get the solution of the integral eq. (18). In [24], Rashed has solved integro-differential equations by using the Chebyshev interpolation method. We also use this method for solving the obtained integral equations. In fact, we apply Chebyshev polynomials of the first kind as the basic functions for approximating the function, q, and convert the integral equations in (18) into systems of linear equations.

Chebyshev polynomials of the first kind $T_k(x)$ on the interval $[-1,1]$ are defined by the recursive relation:

$$T_0(x)=1,\ \ T_1(x)=x,\ \ T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$$

By using Chebyshev interpolation method for the function $q(t)$, one can show that, see [24]:

$$q(t)\cong\sum_{i=0}^{N} q_i l_{i,N}(t), \quad t\in[0,1] \tag{19}$$

where

$$l_{i,N}(t)=\begin{cases}\dfrac{2\delta_i}{N}\sum\limits_{k=0}^{N}{}''T_k(4t-1)\cos\left(\dfrac{ki\pi}{N}\right) & t\in(0,1/2),\\ \dfrac{2\delta_i}{N}\sum\limits_{k=0}^{N}{}''T_k(4t-3)\cos\left(\dfrac{ki\pi}{N}\right) & t\in(1/2,1)\end{cases}$$

$$\delta_i=\begin{cases}0.5, & i=0,N,\\ 1, & 0<i<N\end{cases}$$

the numbers q_i, $i=0,1,...,N$ are the values of the function $q(t)$ in the points $t_i=[\cos(i\pi/N)+1]/4$ and $t_i=[\cos(i\pi/N)+3]/4$ for $t\in(0,1/2)$ and $t\in(1/2,1)$, respectively. The functions $T_k(t)$, $k=0,1,...,N$ Chebyshev polynomials of the first kind and $\sum''$ is the sum of all terms except the first and last two sentences so that the sum of half of the two sentences is considered. Substituting eq. (19) into eq. (18), we obtain:

$$\sum_{i=0}^{N}R(x_n^j,t_i)q_i\cong\mathrm{g}(x_n^j),\;\; j=1,2,...,n-1,\;\; n=2m$$

where

$$R(x_n^j,t_i)q_i\cong\frac{2\delta_i}{N}\sum_{i=0}^{N}{}''I_k(x_n^j)\cos\left(\frac{ki\pi}{N}\right)$$

$$I_k(x_n^j)=\begin{cases}\displaystyle\int_0^{x_n^j}T_k(4t-1)\cos\left(\sqrt{\lambda_n}x_n^j\right)\mathrm{d}t, & x_n^j\in(0,1/2),\\ \displaystyle\int_0^{x_n^j}T_k(4t-3)\left\{\sin\left(\sqrt{\lambda_n}x_n^j\right)+\sin\left[\sqrt{\lambda_n}(1-x_n^j)\right]\right\}\mathrm{d}t- & \\ \displaystyle -2\int_0^{1/2}T_k(4t-3)\sin\left[\sqrt{\lambda_n}\left(1-x_n^j\right)\right]\mathrm{d}t, & x_n^j\in(\tfrac{1}{2},1)\end{cases}$$

$$\mathrm{g}(x_n^j)\cong\begin{cases}2\sqrt{\lambda_n}\sin\left(\sqrt{\lambda_n}x_n^j\right), & x_n^j\in(0,1/2),\\ -2\sqrt{\lambda_n}\left\{\cos\left(\sqrt{\lambda_n}x_n^j\right)-\cos\left[\sqrt{\lambda_n}(1-x_n^j)\right]\right\}- & \\ -4\alpha^+\sin\left(\sqrt{\lambda_n}x_n^j\right)-4\alpha^-\sin\left[\sqrt{\lambda_n}\left(1-x_n^j\right)\right]\mathrm{d}t, & x_n^j\in(1/2,1)\end{cases}$$

Therefore, the solution of inverse nodal problem is calculated by using the following algorithm.

Algorithm. Let the numbers $\{x_n^j\}$, $j=1,2,...,n-1$, $n=2m$, be given.

- Choose N.
- Find the coefficients q_i, $i=0,1,...,N$ by applying the following linear system:

$$A_N\hat{q}\cong B_N,$$

where

$$A_N = [R(x_n^j, t_i)], \quad j = 1,2,...,n-1, \quad n = N+2, \quad i = 0,1,...,N$$

$$B_N = [\mathrm{g}(x_n^j)], \quad j = 1,2,...,n-1, \quad n = N+2$$

$$\hat{q}^T = [q_i], \quad i = 0,1,...,N$$

Numerical examples

In this section, we provide some numerical examples for inverse nodal problem implemented by the given algorithm. We use MATLAB software program for drawing the figures. The convergence of the proposed method and the stability of the inverse problem solution are seen in these examples.

Example 1. Let $q(x) = \exp(x)$ be given. Then, the numerical values of the nodal points of eqs. (1)-(4) obtained from the relations (9) and (10) with $n = 20$ in the intervals $(0,1/2)$ and $(1/2,1)$, respectively, are seen in tab. 1.

Table 1. Numerical values of nodal points in *Example 1*

j	1	2	3	4	5
x_n^j	0.0526000358	0.1052004815	0.1578013592	0.2104026922	0.2630045052
j	6	7	8	9	
x_n^j	0.315606824	0.3682096760	0.4208130901	0.4734170966	
j	10	11	12	13	14
x_n^j	0.5002806681	0.552885624	0.6054912551	0.6580975987	0.7107046930
j	15	16	17	18	19
x_n^j	0.7633125785	0.8159212982	0.8685308970	0.921141422	0.9737529244

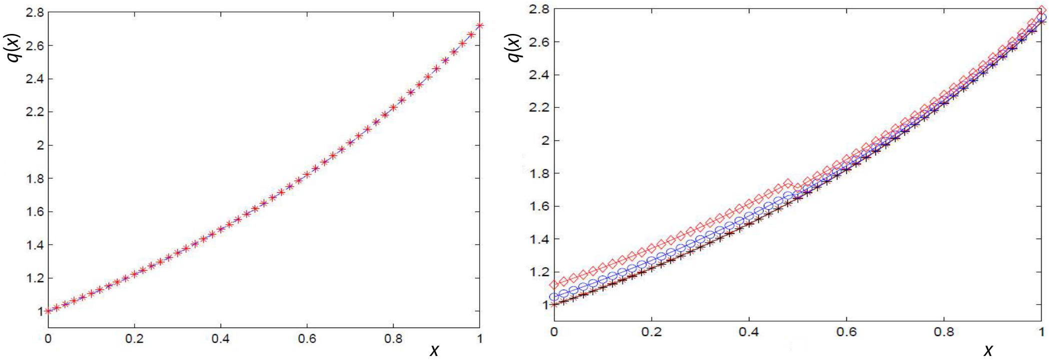
Now, we suppose that q is the unknown function and the nodal points given in tab.1 are the input data. We want to get the approximation of the potential q as the solution of inverse nodal problem by the presented algorithm.

Applying the described algorithm, we obtain the numerical values of the potential function $q(x) = \exp(x)$ with $N = 18$ and calculate the approximate solution of inverse nodal problem by substituting the obtained numerical values into eq. (19). The exact solution and the numerical approximation obtained with $N = 18$ for no noise in the nodal points are seen in fig. 1(a).

Also, we solve inverse nodal problem by using the noisy data $x_n^j(1 + P\varepsilon)$, instead of x_n^j, where P and ε are the amount of noise and the random real number, respectively, in the interval [−1, 1]. The exact solution and the numerical approximations of the function, q, obtained with $N = 18$ and $P = 0, 2\%$ and 5% are seen in fig. 1(b). In fig. 1, it can be shown that the calculated numerical solutions are stable and also become more accurate as the amount of noise P decreases.

Finally, we obtain the absolute errors between the exact and approximate solutions of q for no noise in the nodal points with $N \in \{14,16,18\}$ which are seen in fig. 2. In fig. 2, it can be shown that by increasing the amount of N, the errors are reduced.

Example 2. Let $q(x) = \sin(3\pi x)$ be given. Then, the numerical values of the nodal points of eqs. (1)-(4) obtained from the relations (9) and (10) with $n = 20$ in the intervals $(0,1/2)$ and $(1/2,1)$, respectively, are seen in tab. 2.



(a) (b)

Figure 1. Solution of inverse nodal problem in *Example 1*; (a) exact and approximate solutions of the potential function $q(x) = \exp(x)$ for noise in the nodal points: (----) for approximate solution with $N = 18$ and (***) for exact solution, (b) exact and approximate solutions of the potential function $q(x) = \exp(x)$ with $N = 18$: (◊◊◊) for $p = 5\%$ noise, (ooo) for $p = 2\%$ noise, (---) for $p = 0$ noise and (***) for exact solution

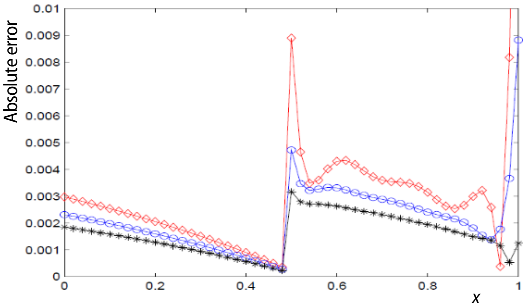


Figure 2. Absolute errors between exact and approximate solutions for no noise in the nodal points in *Example 1*; (◊◊◊) for $N = 14$, (ooo) for $N = 16$, and (***) for $N = 18$

Table 2. Numerical values of nodal points in *Example 2*

j	1	2	3	4	5
x_n^j	0.0525727182	0.1051485930	0.1577264310	0.2103045654	0.2628812580
j	6	7	8	9	
x_n^j	0.3154551181	0.3680254380	0.4205923631	0.4731568573	
j	10	11	12	13	14
x_n^j	0.5001403340	0.5527041708	0.6052697159	0.6578382657	0.7104103925
j	15	16	17	18	19
x_n^j	0.7629858063	0.8155634247	0.8681416338	0.92071868	0.9732930804

Now, we suppose that the nodal points given in tab. 2 are the input data and we get the approximation of the potential q. The exact solution and the numerical approximation obtained with $N = 18$ for no noise in the nodal points and the exact solution and the numerical approximations of the function q obtained with $N = 18$ and $P = 0, 2\%$ and 5% are seen in figs. 3(a) and 3(b).

The absolute errors between the exact and approximate solutions of q for no noise in the nodal points with $N \in \{14, 16, 18\}$ are shown in fig. 4.

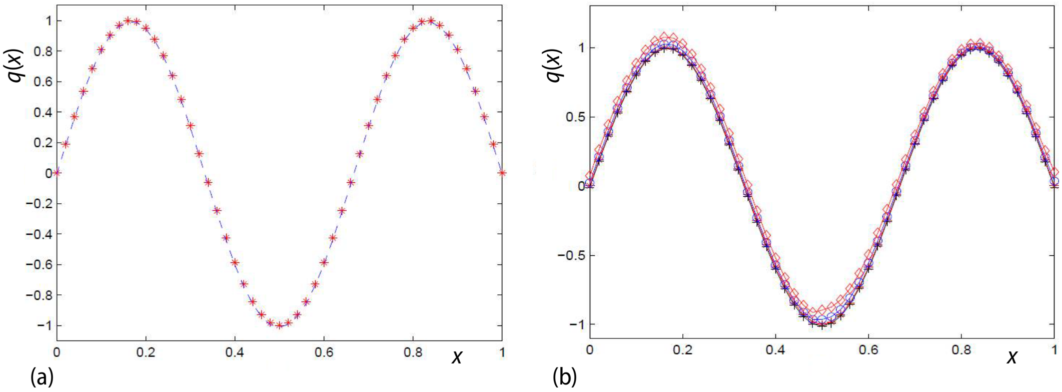


Figure 3. Solution of inverse nodal problem in *Example 2*; (a) exact and approximate solutions of the potential function $q(x) = \sin(3\pi x)$ for no noise in the nodal points: (----) for approximate solution with $N = 18$ and (***) for exact solution, (b) exact and approximate solutions of the potential function $q(x) = \sin(3\pi x)$ with $N = 18$: (◊◊◊) for $p = 5\%$ noise, (ooo) for $p = 2\%$ noise, (---) for $p = 0$ noise, and (***) for exact solution

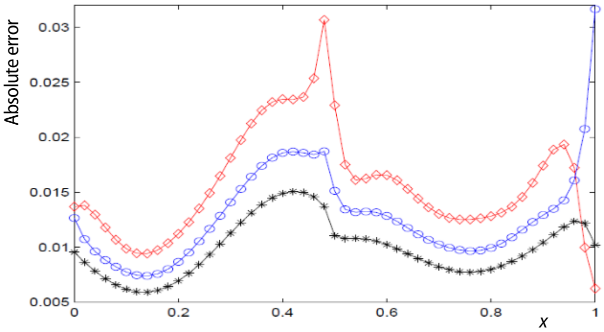


Figure 4. Absolute errors between exact and approximate solutions for no noise in the nodal points in *Example 2*; (◊◊◊) for $N = 14$, (ooo) for $N = 16$, (***) for $N = 18$

Conclusion

In this study, an inverse nodal problem for SL equation with energy dependent jump conditions was solved. Although the problem of eqs. (1)-(4) had jump conditions at $x = 1/2$ we obtained a reconstruction formula for q under a boundless condition. This means that *Theorem 5* (in Section *Some asymptotic expansions for nodal lengths and coefficients*) can not be valid without this condition on q. Then, we showed that this inverse nodal problem was stable. Furthermore, we obtained the approximate solutions of the inverse nodal problem for SL operator with jump conditions by using the Chebyshev interpolation method. We used Chebyshev polynomials of the first kind, and presented the examples for computing the numerical values of the potential function and then showed stable numerical results for these examples. In recent years, Chebyshev interpolation method has frequently been used for solving integral equations and integro-differential equations. In this study, it was shown that this method can be applied for calculating the approximate solution of the inverse SL problem with jump conditions.

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