APPROXIMATE ANALYTICAL SOLUTION FOR 1-D PROBLEMS OF THERMOELASTICITY WITH DIRICHLET CONDITION

by

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This paper presents the solution of the initial boundary-value problem for the system of 1-D thermoelasticity using a new modified decomposition method that takes into accounts both initial and boundary conditions. The obtained solution is based on the generalized form of the inverse operator and is given in the form of a finite series. Also, some numerical experiments were presented to the both the effectiveness and the accuracy of the presented method.

Key words: *thermoelasticity problem, initial-boundary value problem, decomposition method*

Introduction

The domain of thermoelasticity is applied to various problems in physics, and structural and mechanical engineering due to the fact that it gives a deep insight into the nature of the interaction between the elastic and thermal fields [1, 2]. This diversity of applications attracted many authors towards the study of the solution of the system of PDE, often involving various non-linearities. Abd-Alla *et al.* [3] presents two methods for the solution of the 1-D thermoelastic problem for a semi-bounded region (half-space) subjected to prescribed harmonic displacements at the boundary. Both methods used gave a particular solution to the problem not satisfying any thermal boundary conditions. The same system of non-linear coupled PDE, described in [3], and with the same thermal boundary conditions was solved by Rawy *et al.* [4] using the well-known finite difference scheme. Quintanilla [5], analyzed the linearized form of the system of PDE of thermoelasticity. He proved that the solution depends always on the variation of the two thermal constants expressed in the equations. Copetti and French [6] studied the steady-state solutions and the erratic behavior shown in the problem of heat conduction in an elastic rod in contact with a rigid wall using the finite element method. The diversity of applications attracted many authors towards the study of the problem of existence, uniqueness and stability of the system of equation of linear thermoelasticity [7, 8]. Many authors, in recent years, were attracted to apply various numerical techniques for the solution of the thermoelastic problem, such as: variational iteration method [9], explicit and implicit schemes [10], and Adomian decomposition method (ADM) [11, 12]. The ADM is based on splitting the given equation

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into linear and non-linear parts, invert the highest-order derivative operator contained in the linear operator in both sides, calculate Adomian's polynomials, and finally find the successive terms of the series solution by recurrent relation using Adomian polynomials [13-15]. In this paper, we will present a new modification of the ADM with novel structure of the inverse operator applied to the 1-D problem of thermoelasticity with Dirichlet boundary conditions. The operator allows the appearance of all conditions in the solution.

Problem formulation

In this paper, a coupled system of thermoelasticity in [8] is considered in the following form:

$$\omega_{tt}-\frac{1}{x^r}\frac{\partial}{\partial x}\left(x^r\frac{\partial\omega}{\partial x}\right)+\frac{\alpha}{x^r}\frac{\partial}{\partial x}\left(x^r\frac{\partial\theta}{\partial x}\right)=f\left(t,x\right) \tag{1}$$

$$\theta_t-\frac{v}{x^r}\frac{\partial}{\partial x}\left(x^r\frac{\partial\theta}{\partial x}\right)+\beta\omega_t=g\left(t,x\right) \tag{2}$$

in the bounded domain

$$A=\Omega\times(0,T)=\{(x,t):\ a<x<b\ ,\ \ 0<t<T\}$$

where ω is the displacement, θ – the absolute temperature, f – an external force, g – a heat supply, and v,α , and β – are positive real constants and their roles were well explained in [8], where $r = 0, 1, 2$. For $\alpha=\beta=0$, system of eqs. (1) and (2) decouples and two independent equations are obtained, namely the hyperbolic wave equation and the parabolic heat equation *with the Bessel operator*. Both equations have been extensively investigated and several results concerning existence, uniqueness and well-posedness have been established [8]. Equations (1) and (2) are supplemented with the initial conditions:

$$\begin{aligned}&\omega(0,x)=\omega_0(x),\quad a<x<b\\&\omega_t(0,x)=\omega_1(x),\quad a<x<b\\&\theta(0,x)=\theta_0(x),\quad a<x<b\end{aligned} \tag{3}$$

and the Dirichlet boundary conditions:

$$\begin{aligned}&\omega(t,a)=\alpha_1(t),\quad 0<t<T\\&\omega(t,b)=\beta_1(t),\quad 0<t<T\\&\theta(t,a)=\alpha_2(t),\quad 0<t<T\\&\theta(t,b)=\beta_2(t),\quad 0<t<T\end{aligned} \tag{4}$$

Modified ADM for solving Dirichlet problem for 1-D

Thermoelasticity

Several authors used the ADM to solve the initial-boundary value problem based only on the imposition of the initial conditions [16]. However, when initial and boundary conditions have to be imposed the Adomian's series may converge (or not) to a solution with the wrong boundary conditions. Here, it should be mentioned that the sequence of approximate solutions by Adomian's method converges to a certain curve or function. For example, the ADM has been applied by Ebaid [17] to solve the Thomas-Fermi equation which has no exact solution, which is also a real life problem. In that paper, Ebaid showed geometrically that the sequence of the approximate solutions converges to a certain curve which may even be the exact solution for that problem. Hence, a method that takes into accounts both initial and boundary conditions is presented. Our application of the ADM is based on the new inverse operator defined in [17, 18].

The system of eqs. (1)-(4) is considered by re-writing eqs. (1) and (2) in operator form:

$$L_{tt}\omega - L_{xx}\omega + \alpha L_{xx}\theta = f\left(t,x\right) \tag{5}$$

$$L_t\theta - vL_{xx}\theta + \beta L_t\omega = g\left(t,x\right) \tag{6}$$

where

$$L_t = \frac{\partial}{\partial t}\ , \quad L_{tt} = \frac{\partial^2}{\partial t^2}\ , \ \text{ and } \ L_{xx} = \frac{1}{x^r}\frac{\partial}{\partial x}\left(x^r\frac{\partial}{\partial x}\right)$$

To find the solution of problem (1)-(4), firstly, the inverse t operators is defined:

$$L_t^{-1} = \int_0^t (.)\mathrm{d}t, \qquad L_{tt}^{-1} = \int_0^t\int_0^t (.)\mathrm{d}t\mathrm{d}t \tag{7}$$

Applying the inverse operator L_{tt}^{-1} to the both sides of eq. (5) and applying the inverse operator L_t^{-1} to the both sides of eq. (6) we obtain:

$$\omega\left(t,x\right) = \omega\left(0,x\right) + t\omega_t\left(0,x\right) + L_{tt}^{-1}f\left(t,x\right) + L_{tt}^{-1}\left[L_{xx}\omega\right] - \alpha L_{tt}^{-1}\left[L_{xx}\theta\right] \tag{8}$$

$$\theta\left(t,x\right) = \theta\left(0,x\right) + L_t^{-1}g\left(t,x\right) + vL_t^{-1}\left[L_{xx}\theta\right] - \beta L_t^{-1}L_t\omega \tag{9}$$

Then, the inverse x operator L_{xx}^{-1} is defined:

$$L_{xx}^{-1} = \int_a^x \frac{1}{x^r}\mathrm{d}x'\int_0^{x'} x^r\left(.\right)\mathrm{d}x'' - z\left(x\right)\int_a^b \frac{1}{x^r}\mathrm{d}x'\int_0^{x'} x^r\left(.\right)\mathrm{d}x'' \tag{10}$$

where $z(x)$ is defined, as given in [17]:

$$z\left(x\right) = \begin{cases} \dfrac{x-a}{b-a}\ , & r = 0 \\ -1, & r = 1, 2, 3, \ldots \end{cases} \tag{11}$$

For the explanation of the method the following cases will be tested.

The solution of the special case r =0

Applying the inverse operator L_{xx}^{-1} to the both sides of eqs. (8) and (9) we get:

$$L_{xx}^{-1}\omega\left(t,x\right) = L_{xx}^{-1}\left[\omega\left(0,x\right) + t\omega_t\left(0,x\right)\right] + L_{xx}^{-1}L_{tt}^{-1}f\left(t,x\right) + L_{xx}^{-1}L_{tt}^{-1}\left[L_{xx}\omega\right] - \alpha L_{xx}^{-1}L_{tt}^{-1}\left[L_{xx}\theta\right] \tag{12}$$

$$L_{xx}^{-1}\theta\left(t,x\right) = L_{xx}^{-1}\left[\theta\left(0,x\right) + \beta\omega\left(0,x\right)\right] + L_{xx}^{-1}L_t^{-1}g\left(t,x\right) - \beta L_{xx}^{-1}\omega\left(t,x\right) + vL_{xx}^{-1}L_t^{-1}\left[L_{xx}\theta\right] \tag{13}$$

where the inverse operator is defined:

$$L_{xx}^{-1}=\int_a^x \mathrm{d}x'\int_0^{x'}(.)\,\mathrm{d}x''-\frac{x-a}{b-a}\int_a^b \mathrm{d}x'\int_0^{x'}(.)\,\mathrm{d}x''$$

With this definition, we can easily get:

$$L_{xx}^{-1}L_{xx}\,\omega=\omega(t,x)-\omega(t,a)-\frac{x-a}{b-a}\big[\omega(t,b)-\omega(t,a)\big] \tag{14}$$

$$L_{xx}^{-1}L_{xx}\,\theta=\theta(t,x)-\theta(t,a)-\frac{x-a}{b-a}\big[\theta(t,b)-\theta(t,a)\big] \tag{15}$$

Substituting eqs. (14) and (15) into the system of eqs. (12) and (13) gives:

$$\begin{aligned}L_{xx}^{-1}\omega(t,x)=&\,L_{xx}^{-1}\big[\omega(0,x)+t\omega_t(0,x)\big]+L_{xx}^{-1}L_{tt}^{-1}f(t,x)+\\&+L_{tt}^{-1}\left\{\omega(t,x)-\omega(t,a)-\frac{x-a}{b-a}\big[\omega(t,b)-\omega(t,a)\big]\right\}-\\&-\alpha L_{tt}^{-1}\left\{\theta(t,x)-\theta(t,a)-\frac{x-a}{b-a}\big[\theta(t,b)-\theta(t,a)\big]\right\}\end{aligned} \tag{16}$$

$$\begin{aligned}L_{xx}^{-1}\theta(t,x)=&\,L_{xx}^{-1}\big[\theta(0,x)+\beta\omega(0,x)\big]+L_{xx}^{-1}L_t^{-1}g(t,x)-\beta L_{xx}^{-1}\omega(t,x)+\\&+vL_t^{-1}\left[\theta(t,x)-\theta(t,a)-\frac{x-a}{b-a}\big[\theta(t,b)-\theta(t,a)\big]\right]\end{aligned} \tag{17}$$

We re-arrange eqs. (16) and (17) as:

$$\begin{aligned}L_{tt}^{-1}\omega(t,x)=&\,L_{xx}^{-1}\omega(t,x)-L_{xx}^{-1}\big[\omega(0,x)+t\omega_t(0,x)\big]-L_{xx}^{-1}L_{tt}^{-1}f(t,x)+\\&+L_{tt}^{-1}\left\{\omega(t,a)+\frac{x-a}{b-a}\big[\omega(t,b)-\omega(t,a)\big]\right\}+\\&+\alpha L_{tt}^{-1}\left\{\theta(t,x)-\theta(t,a)-\frac{x-a}{b-a}\big[\theta(t,b)-\theta(t,a)\big]\right\}\end{aligned} \tag{18}$$

$$\begin{aligned}L_t^{-1}\theta(t,x)=&\,\frac{1}{v}\big[L_{xx}^{-1}\theta(t,x)\big]-\frac{1}{v}L_{xx}^{-1}\big[\theta(0,x)+\beta\omega(0,x)\big]-\\&-\frac{1}{v}L_{xx}^{-1}L_t^{-1}g(t,x)+\frac{1}{v}\beta L_{xx}^{-1}\big[\omega(t,x)\big]+L_t^{-1}\left\{\theta(t,a)+\frac{x-a}{b-a}\big[\theta(t,b)-\theta(t,a)\big]\right\}\end{aligned} \tag{19}$$

The new system of eqs. (18) and (19) includes all conditions (initial and boundary), but the problem that appears now is the inapplicability for ADM, so we define two functions $u(t,x)$ with the conditions $u(0,x)=u_t(0,x)=0$ such that:

$$\omega(t,x)=\frac{\partial^2 u}{\partial t^2} \tag{20}$$

The other function $v(t,x)$ with the condition $v(0,x)=0$ such that:

$$\theta(t,x)=\frac{\partial v}{\partial t} \tag{21}$$

Substituting eqs. (20) and (21) in the system of eqs. (18) and (19), we obtain:

$$u(x,t)=L_{tt}^{-1}\left\{\omega(t,a)+\frac{x-a}{b-a}\left[\omega(t,b)-\omega(t,a)\right]\right\}-$$
$$-\alpha L_{tt}^{-1}\left\{\theta(t,a)+\frac{x-a}{b-a}\left[\theta(t,b)-\theta(t,a)\right]\right\}-L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]-$$
$$-L_{xx}^{-1}L_{tt}^{-1}f(t,x)+L_{xx}^{-1}u_{tt}+\alpha L_{tt}^{-1}v_t \tag{22}$$

$$v(t,x)=L_t^{-1}\left\{\theta(t,a)+\frac{x-a}{b-a}\left[\theta(t,b)-\theta(t,a)\right]\right\}-\frac{1}{v}L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]-$$
$$-\frac{1}{v}L_{xx}^{-1}L_t^{-1}g(t,x)+\frac{1}{v}L_{xx}^{-1}\left[v_t+\beta u_{tt}\right] \tag{23}$$

As in the standard ADM we define the solution $u(t,x)$ and $v(t,x)$ by the decomposition series:

$$u(t,x)=\sum_{n=0}^{\infty}u_n \quad \text{and} \quad v(t,x)=\sum_{n=0}^{\infty}v_n \tag{24}$$

Substituting eq. (24) into eqs. (22) and (23), we obtain the recurrence relations:

$$u_0(t,x)=L_{tt}^{-1}\left\{\omega(t,a)+\frac{x-a}{b-a}\left[\omega(t,b)-\omega(t,a)\right]\right\}-L_{xx}^{-1}L_{tt}^{-1}f(t,x)-$$
$$-L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]-\alpha L_{tt}^{-1}\left\{\theta(t,a)+\frac{x-a}{b-a}\left[\theta(t,b)-\theta(t,a)\right]\right\} \tag{25}$$

$$v_0=L_t^{-1}\left\{\theta(t,a)+\frac{x-a}{b-a}\left[\theta(t,b)-\theta(t,a)\right]\right\}-\frac{1}{v}L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]-$$
$$-\frac{1}{v}L_{xx}^{-1}L_t^{-1}g(t,x) \tag{26}$$

$$u_{n+1}=L_{xx}^{-1}u_{n_{tt}}+\alpha L_{tt}^{-1}v_{n_t}, \quad n\geq 0 \tag{27}$$

$$v_{n+1}=\frac{1}{v}L_{xx}^{-1}\left(v_{n_t}+\beta u_{n_{tt}}\right), \quad n\geq 0 \tag{28}$$

After calculating the components $u_i, i=0,1,2,\ldots$ and $v_i, i=0,1,2,\ldots$ then substituting eqs. (25)-(28) into eqs. (20) and (21) so the solution is obtained after solving the resulting equations.

The solution of the special case r = 1

Similarly, the inverse operator L_{xx}^{-1} as defined in eq. (10) is applied to both sides of eqs. (8) and (9) to give:

$$L_{xx}^{-1}\omega(t,x)=L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]+L_{xx}^{-1}L_{tt}^{-1}f(t,x)+L_{xx}^{-1}L_{tt}^{-1}\left[L_{xx}\omega\right]-$$
$$-\alpha L_{xx}^{-1}L_{tt}^{-1}\left[L_{xx}\theta\right] \tag{29}$$

$$L_{xx}^{-1}\,\theta(t,x)=L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]+L_{xx}^{-1}L_t^{-1}g(t,x)-\beta L_{xx}^{-1}\omega(t,x)+vL_{xx}^{-1}L_t^{-1}\left[L_{xx}\theta\right] \quad (30)$$

where the inverse operator defined as [17]:

$$L_{xx}^{-1}=\int_a^x\frac{1}{x}\mathrm{d}x'\int_0^{x'}x(.)\mathrm{d}x''+\int_a^b\frac{1}{x}\mathrm{d}x'\int_0^{x'}x(.)\mathrm{d}x''$$

Hence

$$L_{xx}^{-1}L_{xx}\,\omega=\omega(t,x)-2\omega(t,a)+\omega(t,b) \quad (31)$$

$$L_{xx}^{-1}L_{xx}\,\theta=\theta(t,x)-2\theta(t,a)+\theta(t,b) \quad (32)$$

Substituting eqs. (31) and (32) in the system of eqs. (29) and (30) gives:

$$\begin{aligned}L_{xx}^{-1}\omega(t,x)&=L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]+L_{xx}^{-1}L_{tt}^{-1}f(t,x)+\\&+L_{tt}^{-1}\left[\omega(t,x)-2\omega(t,a)+\omega(t,b)\right]-\alpha L_{tt}^{-1}\left[\theta(t,x)-2\theta(t,a)+\theta(t,b)\right]\end{aligned} \quad (33)$$

$$\begin{aligned}L_{xx}^{-1}\,\theta(t,x)&=L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]+L_{xx}^{-1}L_t^{-1}g(t,x)-\beta L_{xx}^{-1}\omega(t,x)+\\&+vL_t^{-1}\left[\theta(t,x)-2\theta(t,a)+\theta(t,b)\right]\end{aligned} \quad (34)$$

We re-arrange eqs. (33) and (34) as:

$$\begin{aligned}L_{tt}^{-1}\omega(t,x)&=L_{xx}^{-1}\omega(t,x)-L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]-L_{xx}^{-1}L_{tt}^{-1}f(t,x)+\\&+L_{tt}^{-1}\left[2\omega(t,a)-\omega(t,b)\right]+\alpha L_{tt}^{-1}\left[\theta(t,x)-2\theta(t,a)+\theta(t,b)\right]\end{aligned} \quad (35)$$

$$\begin{aligned}L_t^{-1}\,\theta(t,x)&=\frac{1}{v}[L_{xx}^{-1}\,\theta(t,x)-L_{xx}^{-1}\theta(0,x)]-\frac{1}{v}L_{xx}^{-1}L_t^{-1}g(t,x)+\\&+\frac{1}{v}\beta L_{xx}^{-1}\left[\omega(t,x)-\omega(0,x)\right]+L_t^{-1}\left[2\theta(t,a)-\theta(t,b)\right]\end{aligned} \quad (36)$$

The new system of eqs. (35) and (36) includes all conditions (initial and boundary), but the problem that appears now is the inapplicability for ADM, so we define two functions $u(t,x)$ with the conditions $u(0,x)=u_t(0,x)=0$ such that:

$$\omega(t,x)=\frac{\partial^2 u}{\partial t^2} \quad (37)$$

The other function $v(t,x)$ with the condition $v(0,x)=0$ such that:

$$\theta(t,x)=\frac{\partial v}{\partial t} \quad (38)$$

Substituting eqs. (37) and (38) in the system of eqs. (35) and (36), we obtain:

$$\begin{aligned}u(x,t)&=L_{tt}^{-1}\left[2\omega(t,a)-\omega(t,b)\right]-\alpha L_{tt}^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-\\&-L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]-L_{xx}^{-1}L_{tt}^{-1}\,f(t,x)+L_{xx}^{-1}u_{tt}+\alpha L_{tt}^{-1}v_t\end{aligned} \quad (39)$$

$$v(t,x)=L_t^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-\frac{1}{v}\,L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]-$$

$$-\frac{1}{v}\,L_{xx}^{-1}L_t^{-1}\,g(t,x)+\frac{1}{v}\,L_{xx}^{-1}\left[v_t+\beta u_{tt}\right] \tag{40}$$

As in the standard ADM defines the solution $u(t,x)$ and $v(t,x)$ by the decomposition series

$$u(t,x)=\sum_{n=0}^{\infty}u_n \quad \text{and} \quad v(t,x)=\sum_{n=0}^{\infty}v_n \tag{41}$$

Substituting eq. (41) into eqs. (39) and (40), we obtain the recurrence relations:

$$u_0(t,x)=L_{tt}^{-1}\left[2\omega(t,a)-\omega(t,b)\right]-\alpha L_{tt}^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-$$

$$-L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]-L_{xx}^{-1}L_{tt}^{-1}\,f(t,x) \tag{42}$$

$$v_0=L_t^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-\frac{1}{v}L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]-\frac{1}{v}\,L_{xx}^{-1}L_t^{-1}g(t,x) \tag{43}$$

$$u_{n+1}=L_{xx}^{-1}u_{n_{tt}}+\alpha L_{tt}^{-1}v_{n_t}, \quad n\geq 0 \tag{44}$$

$$v_{n+1}=\frac{1}{v}L_{xx}^{-1}\left(v_{n_t}+\beta u_{n_{tt}}\right), \quad n\geq 0 \tag{45}$$

After calculating the components u_i, $i=0,1,2,\ldots$ and v_i, $i=0,1,2,\ldots$, then substituting eqs. (42)-(45) into eqs. (37) and (38) the solution is obtained after solving the resulting equations.

The solution of the special case r = 2

With the same procedure as it was used for the special case $r = 1$, the following recurrence relations are obtained:

$$u_0(t,x)=L_{tt}^{-1}\left[2\omega(t,a)-\omega(t,b)\right]-\alpha L_{tt}^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-$$

$$-L_{xx}^{-1}\left[\omega(0,x)+t\omega_t(0,x)\right]-L_{xx}^{-1}L_{tt}^{-1}\,f(t,x) \tag{46}$$

$$v_0=L_t^{-1}\left[2\theta(t,a)-\theta(t,b)\right]-\frac{1}{v}\,L_{xx}^{-1}\left[\theta(0,x)+\beta\omega(0,x)\right]-\frac{1}{v}\,L_{xx}^{-1}L_t^{-1}\,g(t,x) \tag{47}$$

$$u_{n+1}=L_{xx}^{-1}u_{n_{tt}}+\alpha L_{tt}^{-1}v_{n_t}, \quad n\geq 0 \tag{48}$$

$$v_{n+1}=\frac{1}{v}L_{xx}^{-1}\left(v_{n_t}+\beta u_{n_{tt}}\right), \quad n\geq 0 \tag{49}$$

The previous algorithm is used in the next section to solve several examples.

Numerical results and discussion

Example 1. Consider the problem of eqs. (1) and (2) with the value $\alpha=\beta=\gamma=1$ and $f = 2$, $g = -2$.

So, we get the system:

$$\omega_{tt}-\omega_{xx}+\theta_{xx}=2$$

$$\theta_t-\theta_{xx}+\omega_t=-2$$

with the initial conditions

$$\omega(0,x) = x^2, \quad 0 < x < 1$$

$$\omega_t(0,x) = 0, \quad 0 < x < 1$$

$$\theta(0,x) = x^2, \quad 0 < x < 1$$

and the boundary conditions

$$\omega(t,0) = t^2, \quad 0 < t < T$$

$$\theta(t,0) = -t^2, \quad 0 < t < T$$

$$\omega(t,1) = 1 + t^2, \quad 0 < t < T$$

$$\theta\ (t,1) = 1 - t^2, \quad 0 < t < T$$

Applying the proposed method, we obtain the following terms:

$$u_0 := -\frac{1}{12}x^4 + \frac{1}{12}x + \frac{1}{6}t^4 - \frac{1}{2}t^2x^2 + \frac{1}{2}t^2x$$

$$v_0 := -\frac{1}{6}x^4 + \frac{1}{6}x - \frac{1}{3}t^3 + tx^2$$

$$u_1 := \frac{3}{2}t^2x^2 - \frac{1}{12}x^4 + \frac{1}{6}x^3 - t^2x - \frac{1}{12}x - \frac{1}{12}t^4$$

$$v_1 := \frac{1}{2}t^2x^2 + \frac{1}{6}x^3 - \frac{1}{2}t^2x - \frac{1}{6}x$$

$$u_2 := -\frac{1}{2}t^2x^2 + \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}t^2x + \frac{1}{12}x + \frac{1}{6}t^3x^2 - \frac{1}{6}t^3x$$

$$v_2 := \frac{1}{12}x^4t + \frac{1}{4}x^4 - \frac{1}{6}x^3t - \frac{1}{3}x^3 - \frac{1}{2}t^2x^2 + \frac{1}{12}xt + \frac{1}{12}x + \frac{1}{2}t^2x$$

We note that using only ten components the absolute errors becomes zero. In fig. 1 the solution by the proposed method is compared with the exact solution.

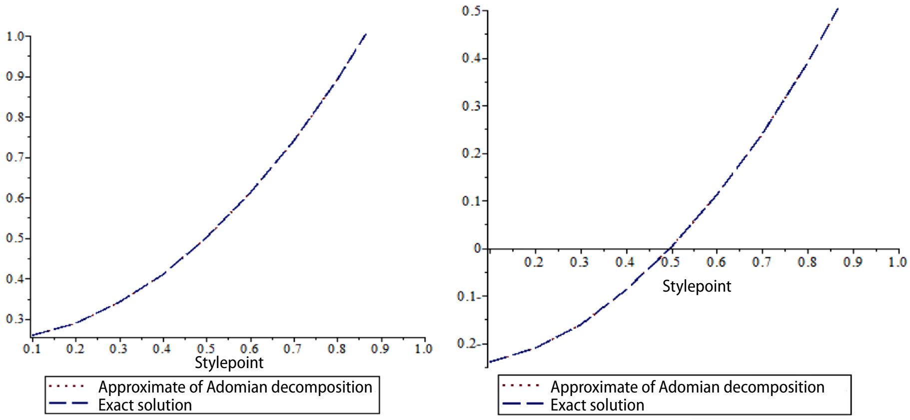

Figure 1. The graphs for the exact and the approximate solutions for *Example 1*

Example 2. Let $\alpha=\beta=\gamma=1$ and $f=\cos(x+t), g=\sin(x+t)$.
Thus, the system takes the form:

$$\omega_{tt}-\omega_{xx}+\theta_{xx}=\cos(x+t)$$

$$\theta_t-\theta_{xx}+\omega_t=\sin(x+t)$$

with the initial conditions

$$\omega(0,x)=\sin x,\quad 0<x<1$$

$$\omega_t(0,x)=\cos x,\quad 0<x<1$$

$$\theta(0,x)=-\cos x,\quad 0<x<1$$

and the boundary conditions

$$\omega(t,0)=\sin t,\quad 0<t<T$$

$$\theta(t,0)=-\cos t,\quad 0<t<T$$

$$\omega(t,1)=\sin(1+t),\quad 0<t<T$$

$$\theta(t,1)=-\cos(1+t),\quad 0<t<T$$

Applying the proposed method, we obtain the following terms:

$$u_0:=\sin(x)+\cos(x)-\sin(t)-\cos(x)\cos(t)+\sin(x)\sin(t)-t\sin(x)+t\cos(x)+$$
$$+x\sin(t)-x\sin(1)\cos(t)-x\cos(1)\sin(t)$$

$$v_0:=x\sin(t)-x\sin(1)\cos(t)-x\cos(1)\sin(t)-\sin(t)+\sin(x)-x\cos(t)+$$
$$+x\cos(1)\cos(t)-x\sin(1)\sin(t)+\cos(t)-\cos(x)\cos(t)+\sin(x)\sin(t)$$

$$u_1:=-1+2x\cos(1)\cos(t)-2x\sin(1)\sin(t)+\sin(x)+\sin(t)+2\cos(t)+x-t+$$
$$+\frac{1}{6}x^3\sin(1)\cos(t)+\frac{1}{6}x^3\cos(1)\sin(t)-\cos(x)\cos(t)+\sin(x)\sin(t)+$$
$$+\frac{5}{6}x\sin(1)\cos(t)+\frac{5}{6}x\cos(1)\sin(t)+t\cos(x)-\frac{1}{6}\sin(t)x^3+xt-2x\cos(t)-$$
$$-\frac{4}{3}x\sin(t)-x\cos(1)-x\sin(1)+\frac{1}{2}\sin(t)x^2-\cos(x)\sin(t)-\sin(x)\cos(t)-$$
$$-x\cos(1)t+x\sin(1)t$$

$$v_1:=\sin(t)+\cos(t)-\frac{2}{3}x\cos(t)-x\sin(t)+\frac{1}{6}\cos(t)x^3+\frac{1}{6}x^3\sin(1)\sin(t)-$$
$$-\frac{1}{6}x^3\cos(1)\cos(t)-\frac{1}{2}\cos(t)x^2-\cos(x)\sin(t)-\sin(x)\cos(t)-\cos(x)\cos(t)+$$
$$+\sin(x)\sin(t)-\frac{7}{6}x\sin(1)\sin(t)+\frac{7}{6}x\cos(1)\cos(t)+x\cos(1)\sin(t)+$$
$$+x\sin(1)\cos(t)$$

$$\begin{aligned}
u_2 :=\ & 1+\frac{1}{3}x\cos(1)\cos(t)-\frac{1}{3}x\sin(1)\sin(t)+\sin(x)-\cos(x)+2\sin(t)-x-t+\\
&+\frac{1}{3}x^3\sin(1)\sin(t)-\frac{1}{3}x^3\cos(1)\cos(t)+\frac{1}{6}x^3\sin(1)+\frac{1}{3}\cos(t)x^3+\\
&+\frac{1}{6}x^3\cos(1)t-\frac{11}{36}x^3\sin(1)\cos(t)-\frac{11}{36}x^3\cos(1)\sin(t)+\frac{1}{120}\sin(t)x^5-\\
&-\cos(t)x^2-\frac{1}{24}\sin(t)x^4+\frac{833}{360}x\sin(1)\cos(t)+\frac{833}{360}x\cos(1)\sin(t)-\\
&\frac{1}{120}x^5\sin(1)\cos(t)-\frac{1}{120}x^5\cos(1)\sin(t)+t\sin(x)+t\cos(x)+\frac{7}{18}\sin(t)x^3+\\
&+\frac{2}{3}xt+\frac{2}{3}x\cos(t)-\frac{61}{45}x\sin(t)+x\cos(1)-\frac{7}{6}x\sin(1)-\sin(t)x^2-\\
&-2\cos(x)\sin(t)-2\sin(x)\cos(t)-\frac{7}{6}x\cos(1)t-x\sin(1)t+\frac{1}{2}x^2t-\frac{1}{6}x^3t
\end{aligned}$$

$$\begin{aligned}
v_2 :=\ & 2\sin(t)+\frac{1}{3}x\cos(t)-\frac{1}{6}x^3\cos(1)\cos(t)+\frac{1}{6}x^3\sin(1)\sin(t)-\sin(t)x^2-\\
&-\frac{1}{2}\cos(t)x^2-\frac{1}{3}x^3\sin(1)\cos(t)-\frac{1}{3}x^3\cos(1)\sin(t)+\frac{1}{6}\cos(t)x^3+\frac{1}{3}\sin(t)x^3-\\
&-2\cos(x)\sin(t)-2\sin(x)\cos(t)-\frac{4}{3}x\sin(t)+\frac{7}{3}x\sin(1)\cos(t)+\\
&+\frac{7}{3}x\cos(1)\sin(t)-\frac{1}{6}x\sin(1)\sin(t)+\frac{1}{6}x\cos(1)\cos(t)
\end{aligned}$$

As it can be seen from tabs.1 and 2, the numerical results of the present method are in very good agreement with their analytical values obtained from the exact solution. In fig. 2 the solution by the proposed method is compared with the exact solution.

Table 1. The absolute error for *Example 2* when $t = 0.5$ in the $\omega(x,t)$

x	ADM solution	Exact solution	Absolute error
0.1	0.56466	0.56464	$1.761660\cdot10^{-5}$
0.2	0.64425	0.64422	$3.350280\cdot10^{-5}$
0.3	0.71740	0.71736	$4.610910\cdot10^{-5}$
0.4	0.78338	0.78333	$5.421040\cdot10^{-5}$
0.5	0.84153	0.84147	$5.698520\cdot10^{-5}$
0.6	0.89126	0.89121	$5.420990\cdot10^{-5}$
0.7	0.93209	0.93204	$4.609200\cdot10^{-5}$
0.8	0.96359	0.96356	$3.348060\cdot10^{-5}$
0.9	0.98547	0.98545	$1.760000\cdot10^{-5}$

Example 3. Let $\alpha=\beta=\gamma=1$ and $f=\mathrm{e}^{x+t}$, $g=-\mathrm{e}^{x-t}$.
Thus, the system takes the form:

$$\omega_{tt}-\omega_{xx}+\theta_{xx}=\mathrm{e}^{x+t}$$

Table 2. The absolute error for *Example 2* when $t = 0.5$ in the $\theta(x,t)$

x	ADM solution	Exact solution	Absolute error
0.1	–0.82533	–0.82534	$1.254900 \cdot 10^{-6}$
0.2	–0.76484	–0.76484	$2.417300 \cdot 10^{-6}$
0.3	–0. 69670	–0.69671	$3.307300 \cdot 10^{-6}$
0.4	–0.62161	–0.62161	$3.880300 \cdot 10^{-6}$
0.5	–0.54030	–0.54030	$4.085900 \cdot 10^{-6}$
0.6	–0.45359	–0.45360	$3.881400 \cdot 10^{-6}$
0.7	–0.36235	–0.36236	$3.304500 \cdot 10^{-6}$
0.8	–0.26750	–0.26750	$2.398600 \cdot 10^{-6}$
0.9	–0.16997	–0.16997	$1.272900 \cdot 10^{-6}$

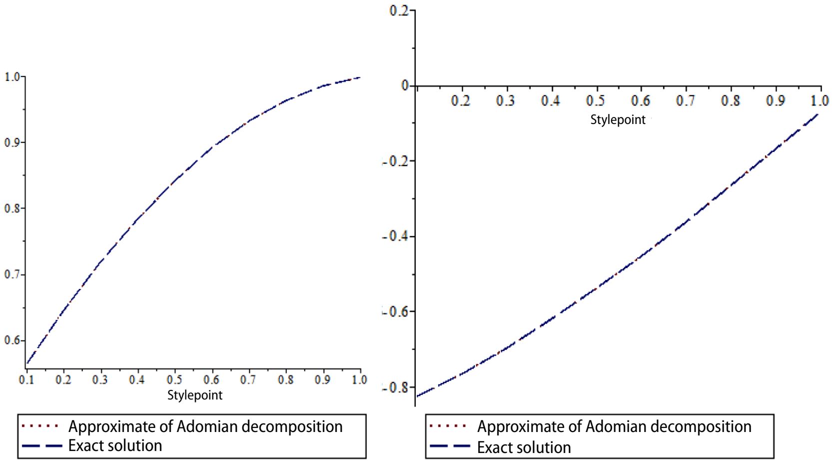

Figure 2. The graphs for the exact and the approximate solutions for *Example 2*

$$\theta_t - \theta_{xx} + \omega_t = -\mathrm{e}^{x-t}$$

with the initial conditions

$$\omega(0,x) = \mathrm{e}^{x}, \quad 0 < x < 1$$

$$\omega_t(0,x) = -\mathrm{e}^{-x}, \quad 0 < x < 1$$

$$\theta(0,x) = \mathrm{e}^{x}, \quad 0 < x < 1$$

and the boundary conditions

$$\omega(t,0) = \mathrm{e}^{-t}, \quad 0 < t < T$$

$$\theta(t,0) = \mathrm{e}^{t}, \quad 0 < t < T$$

$$\omega(t,1) = \mathrm{e}^{1-t}, \quad 0 < t < T$$

$$\theta\ (t,1) = \mathrm{e}^{1+t}, \quad 0 < t < T$$

Applying the proposed method, we obtain the following terms:

$$u_0 := \mathrm{e}^x t - xt\mathrm{e}^{-1} + \frac{t}{\mathrm{e}^x} - \frac{x}{\mathrm{e}^t} + \frac{x\mathrm{e}}{\mathrm{e}^t} + x\mathrm{e}t - \mathrm{e}^x\mathrm{e}^t + \frac{1}{\mathrm{e}^t}$$

$$v_0 := -x\mathrm{e}^t + x\mathrm{e}\mathrm{e}^t + \mathrm{e}^t - \mathrm{e}^x - \frac{x}{\mathrm{e}^t} + \frac{x\mathrm{e}}{\mathrm{e}^t} + \frac{1}{\mathrm{e}^t} - \frac{\mathrm{e}^x}{\mathrm{e}^t}$$

$$u_1 := 2\mathrm{e}^t - \mathrm{e}^x\mathrm{e}^t + \frac{1}{6}\frac{\mathrm{e}x^3}{\mathrm{e}^t} - 2x\mathrm{e}^t - \frac{1}{6}\frac{x^3}{\mathrm{e}^t} + \frac{1}{2}\frac{x^2}{\mathrm{e}^t} + 2x\mathrm{e}\mathrm{e}^t - \frac{7}{6}\frac{x\mathrm{e}}{\mathrm{e}^t} + \frac{2}{3}\frac{x}{\mathrm{e}^t} - \mathrm{e}^x -$$
$$-2x\mathrm{e}t + \mathrm{e}^x t + 2xt - 2t + \frac{\mathrm{e}^x}{\mathrm{e}^t} - \frac{1}{\mathrm{e}^t}$$

$$v_1 := \mathrm{e}^t - \frac{1}{\mathrm{e}^t} + \frac{1}{6}\mathrm{e}\mathrm{e}^t x^3 - \frac{1}{6}\mathrm{e}^t x^3 - \mathrm{e}^x\mathrm{e}^t + \frac{1}{2}\mathrm{e}^t x^2 - \frac{4}{3}x\mathrm{e}^t + \frac{\mathrm{e}^x}{\mathrm{e}^t} + \frac{x}{\mathrm{e}^t} + \frac{5}{6}x\mathrm{e}\mathrm{e}^t - \frac{x\mathrm{e}}{\mathrm{e}^t}$$

$$u_2 := -2 - \frac{7}{36}\frac{\mathrm{e}x^3}{\mathrm{e}^t} + \frac{7}{3}x + \frac{3}{2}x\mathrm{e}\mathrm{e}^t + \frac{1}{120}\frac{\mathrm{e}x^5}{\mathrm{e}^t} - \frac{1}{120}\frac{x^5}{\mathrm{e}^t} + \frac{1}{6}x^3 t - \frac{1}{2}x^2 t -$$
$$-\frac{1}{2}\frac{x^2}{\mathrm{e}^t} - \frac{1}{6}x^3\mathrm{e}t + \frac{16}{45}\frac{x}{\mathrm{e}^t} - 2\mathrm{e}^x\mathrm{e}^t + \frac{67}{360}\frac{x\mathrm{e}}{\mathrm{e}^t} - \frac{1}{2}\mathrm{e}^t x^3 + \frac{3}{2}\mathrm{e}^t x^2 + \frac{1}{9}\frac{x^3}{\mathrm{e}^t} +$$
$$+\frac{1}{3}xt - 3x\mathrm{e}^t + \frac{1}{6}x^3 + \frac{1}{6}x\mathrm{e}t + \frac{1}{2}\mathrm{e}\mathrm{e}^t x^3 - \frac{11}{6}x\mathrm{e} - \frac{1}{6}x^3\mathrm{e} + \frac{1}{24}\frac{x^4}{\mathrm{e}^t} - \frac{1}{2}x^2 +$$
$$+2\mathrm{e}^t + 2\mathrm{e}^x$$

$$v_2 := 2\mathrm{e}^t + \frac{1}{120}\mathrm{e}\mathrm{e}^t x^5 - \frac{1}{120}\mathrm{e}^t x^5 + \frac{17}{36}\mathrm{e}\mathrm{e}^t x^3 + \frac{1}{24}\mathrm{e}^t x^4 + \frac{1}{120}\frac{\mathrm{e}x^5}{\mathrm{e}^t} - \frac{5}{9}\mathrm{e}^t x^3 -$$
$$-\frac{1}{120}\frac{x^5}{\mathrm{e}^t} - 2\mathrm{e}^x\mathrm{e}^t + \frac{3}{2}\mathrm{e}^t x^2 - \frac{1}{36}\frac{\mathrm{e}x^3}{\mathrm{e}^t} + \frac{1}{24}\frac{x^4}{\mathrm{e}^t} - \frac{134}{45}x\mathrm{e}^t - \frac{1}{18}\frac{x^3}{\mathrm{e}^t} + \frac{547}{360}x\mathrm{e}\mathrm{e}^t + \frac{7}{360}\frac{x\mathrm{e}}{\mathrm{e}^t} + \frac{1}{45}\frac{x}{\mathrm{e}^t}$$

As it can be seen from tabs. 3 and 4, the numerical results of the present method are in very good agreement with their analytical values obtained from the exact solution. In fig. 3 the solution by the proposed method is compared with the exact solution.

Example 4. For $\alpha = \beta = \gamma = 1$ and $f = 2, g = -6$ the system takes the form:

$$\omega_{tt} - \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\omega}{\partial x}\right) + \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) = 2$$

$$\theta_t - \frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\theta}{\partial x}\right) + \omega_t = -6$$

with the initial conditions

$$\omega(0,x) = x^2, \quad 0 < x < 1$$

$$\omega_t(0,x) = 0, \quad 0 < x < 1$$

$$\theta(0,x) = x^2, \quad 0 < x < 1$$

Table 3. The absolute error for *Example 3* when $t = 0.5$ in the $\omega(x,t)$

x	ADM solution	Exact solution	Absolute error
0.1	0.67030	0.67032	$1.876510 \cdot 10^{-5}$
0.2	0.74078	0.74082	$3.534820 \cdot 10^{-5}$
0.3	0.81868	0.81873	$4.844000 \cdot 10^{-5}$
0.4	0.90478	0.90484	$5.673140 \cdot 10^{-5}$
0.5	0.99994	1.00000	$6.022250 \cdot 10^{-5}$
0.6	1.10511	1.10517	$5.760400 \cdot 10^{-5}$
0.7	1.22135	1.22140	$4.887700 \cdot 10^{-5}$
0.8	1.34982	1.34986	$3.534800 \cdot 10^{-5}$
0.9	1.49181	1.49182	$1.876500 \cdot 10^{-5}$

Table 4. The absolute error for *Example 3* when $t = 0.5$ in the $\theta(x,t)$

x	ADM solution	Exact solution	Absolute error
0.1	1.82212	1.82212	$2.200000 \cdot 10^{-6}$
0.2	2.01376	2.01375	$3.293000 \cdot 10^{-6}$
0.3	2.22554	2.22554	$4.072000 \cdot 10^{-6}$
0.4	2.45961	2.45960	$2.889000 \cdot 10^{-6}$
0.5	2.71828	2.71828	$4.172000 \cdot 10^{-6}$
0.6	3.00417	3.00417	$2.976000 \cdot 10^{-6}$
0.7	3.32012	3.32012	$4.077000 \cdot 10^{-6}$
0.8	3.66930	3.66930	$3.332000 \cdot 10^{-6}$
0.9	4.05520	4.05520	$2.033000 \cdot 10^{-6}$

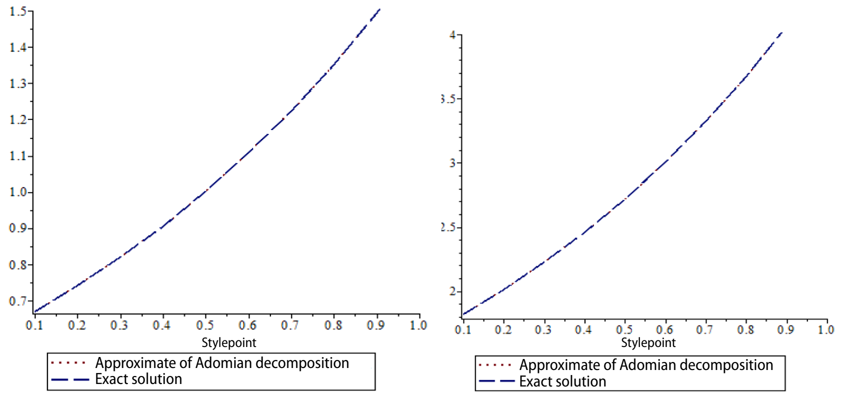

Figure 3. The graphs for the exact and the approximate solutions for *Example 3*

and the boundary conditions

$$\begin{aligned}
&\omega(t,0) = t^2, \quad 0 < t < T \\
&\theta(t,0) = -t^2, \quad 0 < t < T \\
&\omega(t,1) = 1 + t^2, \quad 0 < t < T \\
&\theta(t,1) = 1 - t^2, \quad 0 < t < T
\end{aligned}$$

Applying the proposed method, we obtain the following terms:

$$u_0 := \frac{1}{6}t^4 - \frac{1}{20}x^4 - \frac{1}{20} - \frac{1}{2}\left(\frac{1}{3}x^2 + \frac{1}{3}\right)t^2$$

$$v_0 := -\frac{1}{3}t^3 - \frac{1}{10}x^4 - \frac{1}{10} + x^2 t$$

$$u_1 := -\frac{1}{60}x^4 + \frac{5}{6}t^2x^2 - \frac{1}{18}x^2 - \frac{13}{180} + \frac{1}{3}t^2 - \frac{1}{12}t^4$$

$$v_1 := \frac{1}{30}x^4 + \frac{1}{6}t^2x^2 - \frac{1}{18}x^2 - \frac{1}{45} + \frac{1}{6}t^2$$

$$u_2 := \frac{1}{12}x^4 - \frac{1}{6}t^2x^2 + \frac{1}{9}x^2 + \frac{7}{36} - \frac{1}{6}t^2 + \frac{1}{3}\left(\frac{1}{6}x^2 + \frac{1}{6}\right)t^3$$

$$v_2 := \frac{1}{60}x^4 t + \frac{1}{12}x^4 - \frac{1}{6}t^2x^2 + \frac{1}{18}x^2 t + \frac{1}{9}x^2 + \frac{13}{180}t + \frac{7}{36} - \frac{1}{6}t^2$$

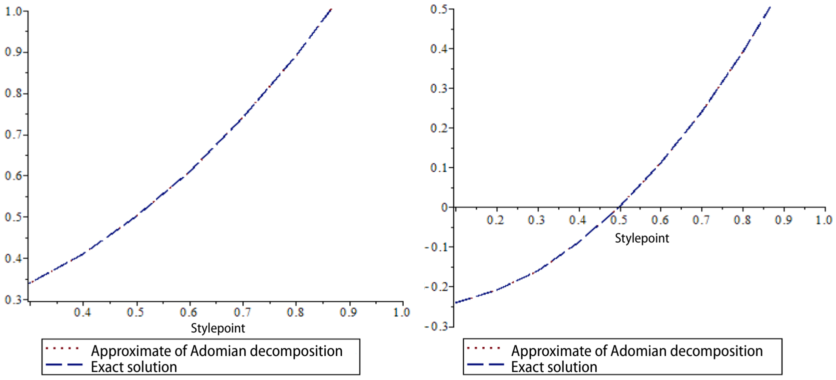

Figure 4. The graphs for the exact and the approximate solutions *Example 4*

We note that using only ten components the absolute errors becomes zero. In fig. 4 the solution by the proposed method is compared with the exact solution.

Conclusions

In this paper, the authors presented a new structure of the inverse operator with the new modified ADM for the solution of the initial-boundary value problems of 1-D thermoelasticty with Dirichlet conditions that takes into accounts both initial and boundary conditions. The proposed new structure proved to be a very effective technique to get solutions of thermoelastic problems with Dirichlet conditions, due to its fast convergence without the need for the discretization of space and time variables. Such conclusions are demonstrated through the shown numerical examples. The advantages of the reported results can be observed from the generalized results that we have obtained. This is because the previous models in the literature were only special cases of our, especially at $r = 0$. However, for non-zero values of this constant r there are no similar models and also no approximate solutions were available.

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