THE METHOD OF SEPARATION OF VARIABLES FOR LOCAL FRACTIONAL KORTEWEG-DE VRIES EQUATION

by

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This paper presents the analytical solution of the local fractional linear Korteweg-de Vries equation in (1 + 1) fractal dimensional space by using the method of separation of variables.

Key words: *Korteweg-de Vries equation, the method of separation of variables, analytical solution, local fractional derivative*

Introduction

Local fractional calculus was used to develop the mathematical models in engineering practice, *e. g.*, heat transfer [1-4], fluid flow [5], film theory [6], electric circuit [7], and others [8-14]. The important model for the shallow water surfaces via local fractional Korteweg-de Vries equation was developed in [5, 15]. The linear form of the local fractional Korteweg-de Vries equation in (1 + 1) fractal dimensional space was written [5]:

$$\frac{\partial^{\vartheta}\Lambda(\psi,\tau)}{\partial\tau^{\vartheta}}+\frac{\partial^{\vartheta}\Lambda(\psi,\tau)}{\partial\psi^{\vartheta}}+\frac{\partial^{3\vartheta}\Lambda(\psi,\tau)}{\partial\psi^{3\vartheta}}=0 \tag{1}$$

where $\partial^{\vartheta}/\partial\tau^{\vartheta}$, $\partial^{\vartheta}/\partial\psi^{\vartheta}$, and $\partial^{3\vartheta}/\partial\psi^{3\vartheta}$ are the local fractional partial derivatives (see section *Preliminary*), and $\Lambda(\psi,\tau)$ describes the wave function of the fractal wave. The method of separation of variables (MOSOS) was developed in [16].

In this article, our aim is to use the MOSOS to handle the local fractional linear Korteweg-de Vries equation in (1 + 1) fractal dimensional space.

Preliminary

The definition and basic formulas of the local fractional derivative (LFD) are given.

Let us set $\mathbb{C}_{\vartheta}(a,b)$ be a set of the non-differentiable function [1, 5].

The LFD of $\Lambda(\psi)$ of order $\vartheta\,(0<\vartheta<1)$ at the point $\psi=\psi_0$ is given by [1-15]:

$$\mathrm{D}^{(\vartheta)}\Lambda(\psi_0)=\frac{\mathrm{d}^{\vartheta}\Lambda(\psi_0)}{\mathrm{d}\psi^{\vartheta}}=\frac{\Delta^{\vartheta}[\Lambda(\psi)-\Lambda(\psi_0)]}{(\psi-\psi_0)^{\vartheta}} \tag{2}$$

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where

$$\Delta^{\vartheta}[\Lambda(\psi)-\Lambda(\psi_0)]\cong\Gamma(1+\vartheta)\Delta[\Lambda(\psi)-\Lambda(\psi_0)] \tag{3}$$

with $\Lambda(\psi)\in\mathbb{C}_{\vartheta}(a,b)$.

For $n\in\mathbb{N}$ and $k\in\mathbb{R}$, some formulas of the LFD of the NF [1, 5] are listed in tab. 1.

Table 1. The basic formulas

NF	LFD
$\dfrac{\psi^{n\vartheta}}{\Gamma(1+n\vartheta)}$	$\dfrac{\psi^{(n-1)\vartheta}}{\Gamma[1+(n-1)\vartheta]}$
$E_{\vartheta}(k\psi^{\vartheta})$	$kE_{\vartheta}(k\psi^{\vartheta})$

Table 2. The NDS of the ordinary differential equations

LFODE	S
$\dfrac{\mathrm{d}^{3\vartheta}\Lambda(\psi)}{\mathrm{d}\psi^{3\vartheta}}+\dfrac{\mathrm{d}^{\vartheta}\Lambda(\psi)}{\mathrm{d}\psi^{\vartheta}}+\kappa\Lambda(\psi)=0$	$\Lambda(\psi)=\sum_{i=1}^{3}\xi_1 E_{\vartheta}(\kappa_i\psi^{\vartheta})$
..	$\Lambda(\psi)=\xi_2 E_{\vartheta}(\kappa_4\psi^{\vartheta})$

For the constants ξ_i (i = 1, 2) and κ_i (i = 1, 2, 3), the non-differentiable solutions (NDS) for the local fractional ordinary differential equations (LFODE) [1] are listed in tab. 2.

Solving the local fractional linear Korteweg-de Vries equation in (1 + 1) fractal dimensional space

Following the idea of the MOSOS [15], we set the solution of eq. (1):

$$\Lambda(\psi,\tau)=\Omega_{\vartheta}(\psi)\Theta_{\vartheta}(\tau) \tag{4}$$

Submitting eq. (4) into eq. (1) gives:

$$\Omega_{\vartheta}(\psi)\frac{\mathrm{d}^{\vartheta}\Theta_{\vartheta}(\tau)}{\mathrm{d}\tau^{\vartheta}}+\Theta_{\vartheta}(\tau)\frac{\mathrm{d}^{\vartheta}\Omega_{\vartheta}(\psi)}{\mathrm{d}\psi^{\vartheta}}+\Theta_{\vartheta}(\tau)\frac{\mathrm{d}^{3\vartheta}\Omega_{\vartheta}(\psi)}{\mathrm{d}\psi^{3\vartheta}}=0 \tag{5}$$

Equation (5) can be written in the form:

$$\frac{\dfrac{\mathrm{d}^{3\vartheta}\Omega_{\vartheta}(\psi)}{\mathrm{d}\psi^{3\vartheta}}+\dfrac{\mathrm{d}^{\vartheta}\Omega_{\vartheta}(\psi)}{\mathrm{d}\psi^{\vartheta}}}{\Omega_{\vartheta}(\psi)}=-\frac{\dfrac{\mathrm{d}^{\vartheta}\Theta_{\vartheta}(\tau)}{\mathrm{d}\tau^{\vartheta}}}{\Theta_{\vartheta}(\tau)} \tag{6}$$

From eq. (6) we obtain the following LFODE:

$$\frac{\mathrm{d}^{3\vartheta}\Omega_{\vartheta}(\psi)}{\mathrm{d}\psi^{3\vartheta}}+\frac{\mathrm{d}^{\vartheta}\Omega_{\vartheta}(\psi)}{\mathrm{d}\psi^{\vartheta}}-\lambda_1\Omega_{\vartheta}(\psi)=0 \tag{7}$$

and

$$\frac{\mathrm{d}^{\vartheta}\Theta_{\vartheta}(\tau)}{\mathrm{d}\tau^{\vartheta}}+\lambda_2\Theta_{\vartheta}(\tau)=0 \tag{8}$$

where both λ_1 and λ_2 (λ_2, $\lambda_1 \neq 0$) are two constants.

In view of eqs. (7) and (8), we have:

$$\Omega_{\vartheta}(\psi)=\sum_{i=1}^{3}\xi_1 E_{\vartheta}(\kappa_i \psi^{\vartheta}) \tag{9}$$

and

$$\Theta_{\vartheta}(\tau)=\xi_2 E_{\vartheta}(\kappa_4 \tau^{\vartheta}) \tag{10}$$

Thus, submitting eqs. (9) and (10) into eq. (4), the solution of eq. (1) can be written in the form:

$$\Lambda(\psi,\tau)=\left[\sum_{i=1}^{3}\xi_1 E_{\vartheta}(\kappa_i \psi^{\vartheta})\right]\xi_2 E_{\vartheta}(\kappa_4 \tau^{\vartheta}) \tag{11}$$

Conclusion

In this work we investigated the local fractional linear Korteweg-de Vries equation in (1 + 1) fractal dimensional space. The MOSOS was used to obtain the non-differentiable solution of the local fractional linear Korteweg-de Vries equation. The obtained result shows the efficiency of the technology for solve the fractal water wave.

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Nomenclature

ψ – space co-ordinate, [m]
ϑ – fractal order, [–]
$\Lambda(\psi,\tau)$ – wave speed, [ms^{-1}]
τ – time, [s]

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