

A NEW INTEGRAL TRANSFORM WITH AN APPLICATION IN HEAT-TRANSFER PROBLEM

by

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In this paper, an new integral transform $J[\phi(\tau)] = 1/\mu \int_0^\infty \phi(\tau) e^{-\mu\tau} d\tau$ is proposed for the first time. The integral transform is used to solve the differential equation arising in heat-transfer problem.

Key words: *heat transfer, integral transform, differential equation, analytical solution*

Introduction

Integral transforms have played important roles in the practical problems involving computational heat and fluid [1]. In mathematics, any transform is expressed by [2]:

$$\Phi(\theta) = \int_{\theta_1}^{\theta_2} \phi(\tau) \Xi(\tau, \theta) d\tau \quad (1)$$

where $\Xi(\tau, \theta)$ is the kernel function, the input of the transform (4) is a function $\Xi(\tau, \theta)$ and the output of the transform (4) is another function $\Phi(\theta)$.

The inverse transform associated inverse kernel, denoted as $\Xi^{-1}(\tau, \theta)$, is suggested as [2]:

$$\phi(\tau) = \int_{\tau_1}^{\tau_2} \Phi(\theta) \Xi^{-1}(\tau, \theta) d\theta \quad (2)$$

In engineering practice, the Laplace transform was used to solve the heat transfer problems in [3, 4]. The Laplace-Carson transform, as a generalized Laplace transform, was considered to handle the heat-exchange problems [5]. We now recall the Laplace and Laplace-Carson transforms.

The Laplace transform of the function $\phi(\tau)$ is defined as [6]:

$$\Phi(s) = L[\phi(\tau)] = \int_0^\infty \phi(\tau) e^{-s\tau} d\tau \quad (3)$$

provided the integral exists for some s , where L is the Laplace transform operator.

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The inverse operator of eq. (3) is written:

$$\phi(\tau) = L^{-1}[\Phi(s)] = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \phi(\tau) e^{s\tau} ds \quad (4)$$

where ω is a real-valued constant.

The Laplace-Carson transform of the function $\phi(\tau)$ is defined as [7]:

$$\Phi(p) = \Pi[\phi(\tau)] = p \int_0^{\infty} \phi(\tau) e^{-p\tau} d\tau \quad (5)$$

provided the integral exists for some p , where Π is the Laplace-Carson transform operator.

The inverse operator of eq. (5) is written:

$$\phi(\tau) = \Pi^{-1}[\Phi(p)] = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \frac{\phi(\tau)}{p} e^{p\tau} dp \quad (6)$$

where ω is a real-valued constant.

A new integral transform

By taking the kernel function $\Xi(\tau, \theta) = e^{-s\tau}$ in eq. (3) into $\Xi(\tau, \theta) = e^{-\mu\tau}/\mu$, a new integral transform of the function $\phi(\tau)$ is defined:

$$\Phi(\mu) = J[\phi(\tau)] = \frac{1}{\mu} \int_0^{\infty} \phi(\tau) e^{-\mu\tau} d\tau \quad (7)$$

provided the integral exists for some μ , where J is the new integral transform operator.

The inverse operator of eq. (7) is defined:

$$\phi(\tau) = J^{-1}[\Phi(\mu)] = \frac{1}{2\pi i} \int_{\omega-i\infty}^{\omega+i\infty} \phi(\tau) \mu e^{\mu\tau} d\mu \quad (8)$$

where ω is a real-valued constant.

The properties of the integral transform are:

(M1) Suppose that $\Phi_1(\mu) = J[\phi_1(\tau)]$ and $\Phi_2(\mu) = J[\phi_2(\tau)]$, then, we have:

$$J[a\phi_1(\tau) + b\phi_2(\tau)] = a\Phi_1(\mu) + b\Phi_2(\mu) \quad (9)$$

where a and b are two constants.

(M2) Suppose that $\Phi(\mu) = J[\phi(c\tau)]$, then, we have:

$$J[\phi(c\tau)] = \frac{1}{c} \Phi\left(\frac{\mu}{c}\right), \quad (10)$$

where c is a constant.

(M3) Suppose that $\Phi(\varpi) = J[\phi(\tau)]$ and the derivative of $\phi(\tau)$ is $\phi^{(1)}(\tau)$. Then, we have:

$$J[\phi^{(1)}(\tau)] = \mu \Phi(\mu) - \frac{\phi(0)}{\mu} \quad (11)$$

(M4) Suppose that $\Phi(\varpi) = J[\phi(\tau)]$ and let the integral of $\phi(\tau)$ is $\int_0^\tau \phi(\tau) d\tau$. Then, we have:

$$J\left[\int_0^\tau \phi(\tau) d\tau\right] = \frac{1}{\mu} \Phi(\mu) \quad (12)$$

Proof

(M1): By using the definition of the integral transform (7), we directly reduce to (M1).

(M2):

$$J[\phi(c\tau)] = \frac{1}{\mu} \int_0^\infty \phi(c\tau) e^{-\mu\tau} d\tau = \frac{1}{c\mu} \int_0^\infty \phi(c\tau) e^{-\frac{\mu c\tau}{c}} d(c\tau) = \frac{1}{c} \Phi\left(\frac{\mu}{c}\right)$$

(M3):

$$J[\phi^{(1)}(\tau)] = \frac{1}{\mu} \int_0^\infty \phi^{(1)}(\tau) e^{-\mu\tau} d\tau = \frac{1}{\mu} [\phi(\tau) e^{-\mu\tau}]_0^\infty + \int_0^\infty \phi(\tau) e^{-\mu\tau} d\tau = \mu \Phi(\mu) - \frac{\phi(0)}{\mu}$$

(M4):

$$J\left[\int_0^\tau \phi(\tau) d\tau\right] = \frac{1}{\mu} \left\{ \left[\int_0^\tau \phi(\tau) d\tau \right] e^{-\mu\tau} \right\}_0^\infty + \frac{1}{\mu^2} \int_0^\infty \phi(\tau) e^{-\mu\tau} d\tau = \frac{1}{\mu} \Phi(\mu)$$

Thus, we finish the proof.

The integral transforms of the functions are given:

$$(V1) \quad J[1] = \frac{1}{\mu^2}$$

$$(V2) \quad J[e^{c\tau}] = \frac{1}{\mu} \int_0^\infty e^{c\tau} e^{-\mu\tau} d\tau = \frac{1}{\mu(\mu - c)}$$

$$(V3) \quad J[\sin(c\tau)] = \frac{c}{\mu(\mu^2 + c^2)}$$

$$(V4) \quad J[\cos(c\tau)] = \frac{1}{\mu^2 + c^2}$$

$$(V5) \quad J[\tau] = \frac{1}{\mu} \int_0^\infty \tau e^{-\mu\tau} d\tau = \frac{1}{\mu^3}$$

Proof

$$J[1] = \frac{1}{\mu} \int_0^{\infty} e^{-\mu\tau} d\tau = \frac{1}{\mu^2} \quad (13)$$

$$J[e^{c\tau}] = \frac{1}{\mu} \int_0^{\infty} e^{c\tau} e^{-\mu\tau} d\tau = \frac{1}{\mu(\mu - c)} \quad (14)$$

$$J[\sin(c\tau)] = \frac{1}{\mu} \int_0^{\infty} \sin(c\tau) e^{-\mu\tau} d\tau = \frac{1}{2i} \left[\frac{1}{\mu(\mu - ci)} - \frac{1}{\mu(\mu + ci)} \right] = \frac{c}{\mu(\mu^2 + c^2)} \quad (15)$$

$$J[\cos(c\tau)] = \frac{1}{\mu} \int_0^{\infty} \cos(c\tau) e^{-\mu\tau} d\tau = \frac{1}{2} \left[\frac{1}{\mu(\mu - ci)} + \frac{1}{\mu(\mu + ci)} \right] = \frac{1}{\mu^2 + c^2} \quad (16)$$

$$J[\tau] = \frac{1}{\mu} \int_0^{\infty} \tau e^{-\mu\tau} d\tau = \frac{1}{\mu^3} \quad (17)$$

Thus, the proof is finished.

Solving the heat-transfer problem

Let us consider the differential equation in heat-transfer problem [8, 9]:

$$-hM\Theta(x) = \rho V c_p \Theta^{(1)}(x) \quad (18)$$

subject to the initial condition:

$$\Theta(0) = \beta \quad (19)$$

where $\Theta(x)$ is the temperature.

Taking the integral transform of eq. (18) gives:

$$-hM\Theta(\mu) = \rho V c_p \left[\mu \Theta(\mu) - \frac{\beta}{\mu} \right] \quad (20)$$

From eq. (20), we have:

$$\Theta(\mu) = \frac{\beta}{\mu \left(\mu + \frac{hM}{\rho V c_p} \right)} \quad (21)$$

Thus, by taking the inverse transform of eq. (21), the solution of eq. (13) takes the form:

$$\Theta(x) = \beta e^{-\frac{hM}{\rho V c_p} x} \quad (22)$$

which is in agreement with the result in [9].

Conclusion

A new integral transform based on the kernel function $\Xi(\tau, \theta) = e^{-\mu\tau}/\mu$ was presented for the first time. The analytical solution of the differential equation involving the heat-transfer was obtained. The technology is proposed, as a powerful approach, to solve the differential equations.

Nomenclature

c_p – specific heat of the material, [$\text{Jkg}^{-1}\text{K}^{-1}$]
 h – convection heat transfer coefficient, [$\text{Wm}^{-2}\text{K}^{-1}$]
 M – surface area of the body, [m^2]
 V – volume, [m^3]
 x – space co-ordinate, [m]

Greek symbols

$\Theta(x)$ – temperature, [Km^{-3}]
 ρ – density, [kgm^{-3}]

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