SOLUTIONS OF CATTANEO-HRISTOV MODEL OF ELASTIC HEAT DIFFUSION WITH CAPUTO-FABRIZIO AND ATANGANA-BALEANU FRACTIONAL DERIVATIVES

by

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Recently Hristov using the concept of a relaxation kernel with no singularity developed a new model of elastic heat diffusion equation based on the Caputo-Fabrizio fractional derivative as an extended version of Cattaneo model of heat diffusion equation. In the present article, we solve exactly the Cattaneo-Hristov model and extend it by the concept of a derivative with non-local and non-singular kernel by using the new Atangana-Baleanu derivative. The Cattaneo-Hristov model with the extended derivative is solved analytically with the Laplace transform, and numerically using the Crank-Nicholson scheme.

Key words: *Atangana-Baleanu derivatives, numerical approximation, Cattaneo-Hristov model, elastic media*

Introduction

Diffusion phenomena, of heat or mass, are generally described as a consequence of the conservation law by the relationship (assuming constant transport properties of the medium):

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}, \quad \rho = \text{constant}, \quad C_p = \text{constant} \tag{1}$$

With the assumption that $q(x,t) = -k\,\partial T(x,t)/\partial x$, eq. (1) leads to the Fourier (Fick) law thus defining an infinite speed of propagation of the flux which is unphysical. The Cattaneo concept [1, 2] of finite speed of heat diffusion in rigid conductors generalizes the Fourier law by a linear superposition of the heat flux and its time derivative related to its history, namely:

$$q(x,t) = -\int_{-\infty}^{t} R(x,t)\nabla T(x,t-s)\,\mathrm{d}s \tag{2}$$

In case of space-independent damping function $R(x,t)$ it can be represented by the Jeffrey kernel [2] $R(t) = \exp[-(t-s)/\tau]$ with a finite relaxation time $\tau = \text{constant}$ that finally yields the Cattaneo equations [2]:

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$$\frac{\partial T(x,t)}{\partial t} = -\frac{a_2}{\tau}\int_0^t \exp\left[-\left(\frac{t-s}{\tau}\right)\right]\frac{\partial T(x,s)}{\partial x}\mathrm{d}s, \quad a_2 = \frac{k_2}{\rho C_p} \tag{3}$$

With eq. (3), the energy conservation equation of the internal energy [1-4] results in the Jeffrey type integro-differential equation [5-8]:

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{a_2}{\tau}\int_{-\infty}^t \mathrm{e}^{-\left(\frac{t-s}{\tau}\right)}\frac{\partial^2 T(x,s)}{\partial x^2}\mathrm{d}s, \quad a_1 = \frac{k_1}{\rho C_p}, \quad a_2 = \frac{k_2}{\rho C_p} \tag{4}$$

This integro-differential equation can be developed further in two directions;
– integro-differential models allowing deep investigations of the role of the damping kernel on the behavior of the heat diffusion process, and
– the well-known telegraph equation, as it was done in the classical work of Cattaneo [9]

The present work addresses the model, eq. (4), especially its second term known as elastic part of the heat diffusion [1-3, 8], expressed:

$$\frac{\partial T(x,t)}{\partial t} = \frac{a_2}{\tau}\int_{-\infty}^t \mathrm{e}^{-\left(\frac{t-s}{\tau}\right)}\frac{\partial^2 T(x,s)}{\partial x^2}\mathrm{d}s \tag{5}$$

The model of heat has been study in the literature by many researchers [1-8, 10]. It was solved approximately [3, 8] by the integral balance method usefully applicable to time fractional diffusion equations [9].

Recently, the Caputo-Fabrizio (CF) fractional derivative with a non-singular kernel [11-13]:

$${}_{\mathrm{CF}}\mathrm{D}_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}s \tag{6}$$

was defined to model relaxation phenomena in complex media.

Denoting $1/\tau = (1-\alpha)/\alpha \in [0,\infty]$, where $0 < \alpha < 1$, as well as with $M(\alpha) = 1$ it is possible to express eq. (5):

$$\frac{\partial T(x,t)}{\partial t} = a_2(1-\alpha)\,{}_{\mathrm{CF}}\mathrm{D}_t^{\alpha}\frac{\partial^2 T(x,t)}{\partial x^2} \tag{7a}$$

Further, applying the solution developed in [8] the complete heat transfer equation in terms of Caputo-Fabrizio derivatives is:

$$\frac{\partial T(x,t)}{\partial t} = a_1\frac{\partial^2 T(x,t)}{\partial x^2} + a_2(1-\alpha)\,{}_{\mathrm{CF}}\mathrm{D}_t^{\alpha}\frac{\partial^2 T(x,t)}{\partial x^2} \tag{7b}$$

Equation (7b) is the complete Cattaneo-Hristov equation (CHE) of transient heat diffusion equation developed recently [8]. We shall mention that many other good mathematical studies using alternative fractional derivatives have been developed [14-16] but their applications to the present model are beyond the scope of the present work.

Exact solution of CHE based on Caputo-Fabrizio fractional derivative

In this section, we present the solution of the CHE based on the Caputo-Fabrizio derivative. To obtain this solution we using the method of separation of variable combined with the Laplace transform method. We assume that the solution of equation (7a) is given as $T(x,t)=T_1(x)T_2(t)$ then replacing this in equation (7a), we obtain:

$$\frac{\partial[T_1(x)T_2(t)]}{\partial t}=a_2(1-\alpha)\,{}_{\mathrm{CF}}\mathrm{D}_t^{\alpha}\,\frac{\partial^2[T_1(x)T_2(t)]}{\partial x^2} \tag{7c}$$

Rearranging, we obtain the following set of ordinary differential equations:

$$\begin{cases}\dfrac{\mathrm{d}T_1(x)}{\mathrm{d}x}+\lambda^2(1-\alpha)\dfrac{\mathrm{d}^2T_1(x)}{\mathrm{d}x^2}=0\\[2mm] \dfrac{\mathrm{d}T_2(t)}{\mathrm{d}t}+\lambda^2\,{}_{\mathrm{CF}}\mathrm{D}_t^{\alpha}T_2(t)=0\end{cases} \tag{7d}$$

The second equation is solved via the Laplace transform operator. Thus, applying the Laplace transform on both sides, we obtain:

$$\begin{gathered}pT_2(p)-T_2(0)-\lambda^2\frac{pT_2(p)-T_2(0)}{p+(1-p)\alpha}=0\\ T_2(p)\left[p-\frac{p\lambda^2}{p+(1-p)\alpha}\right]=T_2(0)\left[1-\frac{\lambda^2}{p+(1-p)\alpha}\right]\\ T_2(p)=T_2(0)\frac{\left[1-\dfrac{\lambda^2}{p+(1-p)\alpha}\right]}{\left[p-\dfrac{p\lambda^2}{p+(1-p)\alpha}\right]}\\ T_2(p)=\frac{T_2(0)}{p}\end{gathered} \tag{7e,f,g,h}$$

Applying the inverse Laplace on equation (7h), we obtain:

$$T_2(t)=T_2(0) \tag{7i}$$

The solution of the second equation is given:

$$T_1(t)=T_1(0)\exp\left[-\frac{\lambda x}{\sqrt{a_1}(1-\alpha)}\right]+T_1^{'}(0)\exp\left[\frac{\lambda x}{\sqrt{a_1}(1-\alpha)}\right] \tag{7j}$$

Therefore, the exact solution of the CHE based on the Caputo-Fabrizio derivative is given:

$$T(x,t)=\sum_{n=0}^{\infty}T_2(0)\left\{T_1(0)\exp\left[-\frac{\lambda_n x}{\sqrt{a_1}(1-\alpha)}\right]+T_1^{'}(0)\exp\left[\frac{\lambda_n x}{\sqrt{a_1}(1-\alpha)}\right]\right\} \tag{7k}$$

Relaxation kernel through the Mittag-Leffler function and the Atangana-Baleanu derivative

As a step of the development of the CHE, Hristov [8] has defined a general form of the relaxation term with exponential kernel termed as a pro-Caputo (PC) non-normalized derivative ${}_{\mathrm{PC}}\mathrm{D}_t^{\beta}$, namely:

$$ {}_{\mathrm{PC}}\mathrm{D}_t^{\beta} F(x,t) = \beta \int_{-\infty}^{t} \mathrm{e}^{-\beta(t-s)} \frac{\mathrm{d}F(x,s)}{\mathrm{d}t} \mathrm{d}s, \quad \beta = \frac{1}{\tau} \tag{8a,b} $$

In the case of heat conduction $F(x,t) = \partial^2 T(x,t)/\partial x^2$ that leads to eq. (7a). The elastic part of the heat-conduction eq. (5) in this case can be expressed:

$$ \frac{\partial T(x,t)}{\partial t} = -a_2 \frac{\alpha}{1-\alpha} \, {}_{\mathrm{PC}}\mathrm{D}_t^{\alpha} \left[\frac{\partial T(x,s)}{\partial x} \right] \tag{9} $$

Now, we address a kernel defined through the Mittag-Leffler function:

$$ E_\gamma = \sum_{0}^{\infty} \frac{z^k}{\Gamma(\gamma k + 1)} $$

which is a generalization of a family of relaxing functions. The exponential function used in the Jeffrey kernel is a special case of E_γ for $\gamma = 1$.

The eq. (9) is based upon the concept of exponential decay law, which is a particular case of the generalized Mittag-Leffler function. It was also revealed in the latex development of fractional differentiation that, the derivative proposed by Caputo and Fabrizio was a filter not a fractional derivative based upon the fact that the kernel used is local and may not be able to portray more accurately the complex system via which the flow of heat is taking please. The concept of elasticity may perhaps be described with a kernel with non-local design, which is the case of the generalized Mittag-Leffler function. In this section the CHE will be extended to the concept of non-local and non-singular kernel.

The Atangana-Baleanu factional derivative and the associated fractional integral are defined [16, 17]:

$$ {}^{\mathrm{ABR}}_{\quad\mathrm{a}}\mathrm{D}_t^{\alpha}[f(t)] = \frac{B(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_{a}^{t} f(x) E_\alpha \left[-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] \mathrm{d}x \tag{10a} $$

$$ {}^{\mathrm{AB}}_{\ \mathrm{a}}I_t^{\alpha} \left[\mathrm{d}(t) \right] = \frac{1-\alpha}{B(\alpha)} \mathrm{d}(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{a}^{t} \mathrm{d}(j)(t-j)^{\alpha-1} \mathrm{d}j \tag{10b} $$

where $f \in H^1(a,b),\ b > a,\ \alpha \in [0,1]$ and the normalization function $B(\alpha)$ has the same properties as in Caputo and Fabrizio case previously commented. With the Atangana-Baleanu derivative the elastic part of the Cattaneo-Hristov equation can be presented that:

$$ \begin{gathered} \frac{\partial T(x,t)}{\partial t} = a_2 (1-\alpha) \, {}^{\mathrm{ABC}}_{\mathrm{PAB}}\mathrm{D}_t^{\alpha} \left[\frac{\partial T(x,t)}{\partial x} \right] \\ {}^{\mathrm{ABC}}_{\mathrm{PAB}}\mathrm{D}_t^{\alpha} \left[\frac{\partial T(x,t)}{\partial x} \right] = B(\alpha) \int_{0}^{t} \frac{\partial^2 T(x,y)}{\partial y \partial x} E_\alpha \left[-\frac{\alpha}{1-\alpha} (t-y)^\alpha \right] \mathrm{d}y \end{gathered} \tag{11a,b} $$

Exact solution of CHM based on Atangana-Baleanu fractional derivative

Again, in this section eq. (11a) will be solved via method Laplace transform operator. In this case we assume that the initial conditions at $t = 0$ are null. Therefore, applying the Laplace transform on both sides of the eq. (11a) we obtain:

$$pT(x,p)-T(x,0)=\frac{p^{\alpha}\dfrac{\partial T(x,p)}{\partial x}-p^{\alpha-1}\dfrac{\partial T(x,0)}{\partial x}}{p^{\alpha}+\dfrac{\alpha}{1-\alpha}}$$

$$pT(x,p)=\frac{p^{\alpha}\dfrac{\partial T(x,p)}{\partial x}}{p^{\alpha}+\dfrac{\alpha}{1-\alpha}} \tag{11c}$$

$$\frac{\dfrac{\mathrm{d}T(x,p)}{\mathrm{d}x}}{T(x,p)}=\frac{p^{\alpha}+\dfrac{\alpha}{1-\alpha}}{p^{\alpha-1}}$$

$$T(x,p)=\exp\left(x\frac{p^{\alpha}+\dfrac{\alpha}{1-\alpha}}{p^{\alpha-1}}\right)=\sum_{k=0}^{\infty}\frac{x^k p^k}{k!}\sum_{j=0}^{\infty}\frac{\left(x\dfrac{\alpha}{1-\alpha}\dfrac{1}{p^{\alpha-1}}\right)^j}{j!}\frac{1}{p^{\alpha j-j}}=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^{k+j}}{k!j!}\frac{1}{p^{\alpha j-j-k}}$$

Therefore, taking the inverse Laplace operator, we obtain:

$$T(x,t)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^{j+k}}{j!k!}\frac{t^{\alpha j-j-k-1}}{\Gamma\left(\alpha j-j-k\right)} \tag{11d}$$

Numerical solution of the Cattaneo-Hristov model with Atangana-Baleanu fractional derivative

The Crank-Nicholson approximation of time component:

$$\frac{T_i^{n+1}-T_i^n}{\Delta t}=-a_1\frac{B(\alpha)}{1-\alpha}\sum_{k=1}^{n}\frac{\partial_x T_i^{n+1}-\partial_x T_i^n}{\Delta t}\delta_{i,n}$$

$$\delta_{i,n}=\int_{t_k}^{t_{k+1}}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}\left(t_n-y\right)^{\alpha}\right]\mathrm{d}y \tag{12}$$

The previous equation can be rearranged:

$$\frac{T_i^{n+1}}{\Delta t}=\frac{T_i^n}{\Delta t}-a_1\frac{B(\alpha)}{1-\alpha}\sum_{k=1}^{n}\frac{\partial_x T_i^{n+1}-\partial_x T_i^n}{\Delta t}\int_{t_k}^{t_{k+1}}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}\left(t_n-y\right)^{\alpha}\right] \tag{13}$$

where the integral is evaluated:

$$\int_{t_k}^{t_{k+1}}E_{\alpha}\left[-\frac{\alpha}{1-\alpha}(t_n-y)^{\alpha}\right]\mathrm{d}y=(t_n-t_{k+1})E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(t_n-t_{k+1})\right]-(t_n-t_k)E_{\alpha,2}\left[-\frac{\alpha}{1-\alpha}(t_n-t_k)\right]$$

Using the numerical scheme, we generate the following numerical representation, figs. 1-4. It is very import to notice that the flow of heat via elastic medium as is described by CHM model depends also of the fractional order derivative. Therefore, one can conclude that, the fractional order takes account the variation of the elastic medium each fractional order represents a specific property of elasticity.

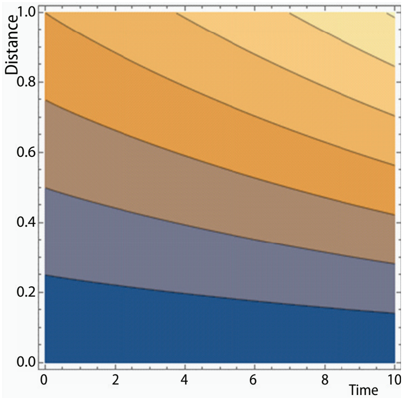

Figure 1. Numerical simulation for $\alpha = 0.95$

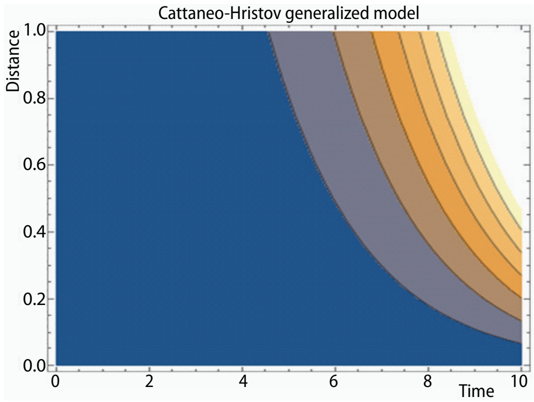

Figure 2. Numerical simulation for $\alpha = 0.65$

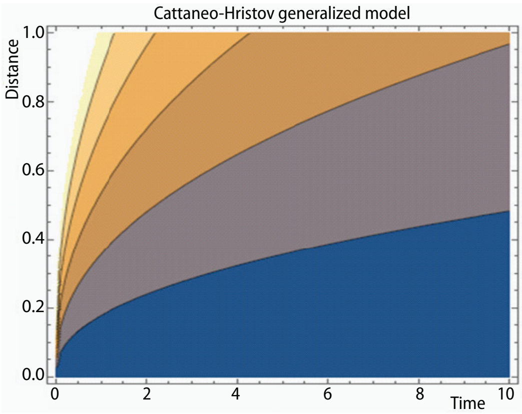

Figure 3. Numerical simulation for $\alpha = 0.35$

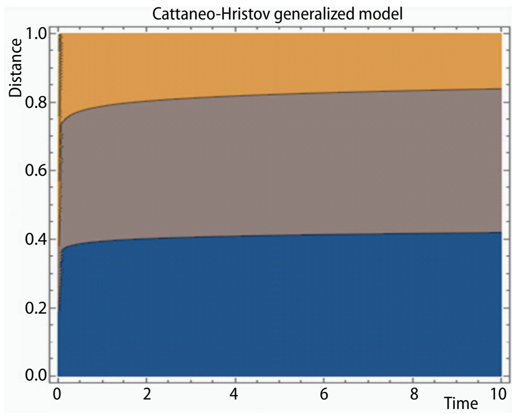

Figure 4. Numerical simulation for $\alpha = 0.05$

Conclusion

The Cattaneo-Hristov elastic heat conduction equation was solved exactly analytically and then extended by a new fractional derivative with a kernel based on the Mittag-Leffler kernel. There is no numerical version of these derivatives in the literature. In this paper, the numerical approximation for first and second order approximation was presented. To apply this new approximation, we generalized the Cattaneo-Hristov model of elastic heat diffusion and solved it using explicit difference scheme and analytical methods based on Laplace transform operator.

Nomenclature

T – temperature, [°C]
x – space, [m]
t – time, [s]

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