SPECIFIC ASPECTS OF TURBULENT FLOW IN RECTANGULAR DUCTS

by

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The essential ideas of investigations of turbulent flow in a straight rectangular duct are chronologically presented. Fundamentally significant experimental and theoretical studies for mathematical modeling and numerical computations of this flow configuration are analyzed. An important physical aspect of this type of flow is presence of secondary motion in the plane perpendicular to the streamwise direction, which is of interest from both the engineering and the scientific viewpoints. The key facts for a task of turbulence modeling and optimal choice of the turbulence model are obtained through careful examination of physical mechanisms that generate secondary flows.

Key words: *turbulent flow, rectangular duct, secondary flows, symmetry, opposite pairs, driving mechanisms, turbulence models*

Introduction

Secondary flows – fluid motions which occur in turbulent flows in straight ducts of non-circular cross-section (Prandtl's second kind of secondary flows) and in curved pipes of any cross-section (Prandtl's first kind of secondary flows, which additionally exist in laminar flow regime) in a plane perpendicular to the direction of main flow – are of considerable engineering interest. Many examples which occur in engineering practice, *e. g.* in components of fluid machinery and heat transfer equipment, include flows in: heat exchangers (compact heat exchangers, recuperators, inter-coolers), turbomachinery impellers and blade passages, air intake ducts of jet engines, diffusers, ventilation and air-conditioning systems, electrostatic precipitators of thermal power plants, and cooling systems in: gas turbines, rocket combustion chambers, nuclear-reactors [1]. Secondary circulation is an important phenomenon in natural streams as well, *e. g.* sediment transport in rivers and canals [2]. The impact of duct corners on the flow friction and heat transfer characteristics arising from different non-circular geometries is thoroughly analyzed in order to gain further insights into the mechanisms responsible for the coexistence of laminar and turbulent flows, bearing in mind the reduction of fluid flow losses and electronic cooling applications [3, 4]. Furthermore, it is noteworthy to develop the understanding of fluid dynamics [5] and heat transfer [6] behavior of non-Newtonian fluids

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flowing through non-circular ducts encountered in the chemical, pharmaceutical, biological and food industries. It is important to emphasize the common features of and the links between three seemingly different internal flows, turbulent duct flow of Newtonian fluids, and tube flows of viscoelastic and particle laden fluids in order to help bridge the mechanics of all three, because secondary motions appear as a common thread through afore-mentioned flow types. It is not possible to develop a good understanding of the mechanics of secondary field in the flow of viscoelastic and particle laden fluids without a clear grasp of the underlying driving mechanisms of the turbulent secondary flows of Newtonian fluids [7]. At the end of this introductory paragraph on secondary motions from a broad viewpoint (fluid – geometry – flow; fluid stress-strain rate relationship: Newtonian – non-Newtonian; duct centerline axis: straight – curved; duct cross-sectional shape: circular – non-circular; duct cross-sectional area/shape: constant – varying; flow configuration: external – internal; flow regime: laminar – turbulent), there is an interesting fact observed in corner-flow configurations of Newtonian fluids where we encounter one of the rare cases where laminar and turbulent flows show completely different tendencies in their general behavior. It appears that the secondary flow along the bisector of the corner is directed away from the corner in laminar flow and towards the corner in turbulent flow, although the driving mechanisms in these cases are different, contrary to previously mentioned similarities in causes of secondary motion in Newtonian and non-Newtonian fluid flows [8].

The various effects caused by turbulent secondary flows can have profound consequences for engineering design and calculation methods. Secondary flow exerts a significant influence on transport phenomena (momentum, heat, and mass transfer) in turbulent flows and it is this factor which determines considerable interest in this phenomenon [9]. The flow through a straight duct of rectangular cross-section has important similarities in common with both straight circular pipe flow and plane channel flow (well-known canonical flows). However, geometrical characteristics of its cross-sectional shape (rectangle is not perfectly symmetrical – the number of symmetry lines is not infinite, contrary to the ideal shape of circle; moreover, rectangle has two additional sides/walls in comparison with two infinite parallel plates/lines) in synergy with turbulence lead to the remarkable changes of the mean flow field characteristics and turbulent flow structure. Apart from the deformation of primary velocity contours, the location of velocity maximum is shifted vertically downward from the free surface in open channels. In addition, the friction behavior is strongly affected by variation of the local wall shear-stress along the duct walls [10]. In a similar way, the local wall heat flux [11] and the particle transport [12] are significantly influenced by secondary motions.

The goal of every engineer is to always improve the efficiency and performance of the machines and equipment that are to be designed. In order to do so, it is important to understand the phenomena and processes taking place in their components. Furthermore, certain important engineering problems, like the study of behavior and origin of secondary currents, arising in turbulent flow in a straight rectangular duct, demand a deeper insight into the mechanism of the flow. A satisfactory description and understanding of the mechanism, origin and prediction of Prandtl's second kind of secondary flows have been an outstanding issue for many decades, despite continuous efforts from a number of researchers. This problem has been studied from many points of view, through experiments, theoretical analyses and different numerical approaches: turbulence modeling and numerical computations of the Reynolds-averaged Navier-Stokes equations (RANS), large-eddy simulations (LES), and direct numerical simulations (DNS). From the scientific viewpoint the nature and cause of secondary flows is still somewhat intriguing, due to ambiguous and unexpected role of turbulence, as one of the fundamental fluid mechanics problems. On the one hand turbulence tends to reduce the secondary flow of the first kind, and on the other hand it represents a generating mechanism for the mean secondary flow of the second kind – an organized, rather than chaotic, cellular flow pattern perpendicular to the main channel flow is characterized by a symmetrical system of paired counter-rotating vortical structures bounded by the walls of the duct and bisectors of the walls and corners. Therefore, the understanding of basic mechanisms of turbulent flow is vital for modeling and reliable numerical predictions of turbulence-driven secondary motions [13-15].

Detection and classification of secondary flows

Prandtl and Nikuradse were among the first researchers to experimentally investigate the flow characteristics in straight ducts of uniform but non-circular cross-section, fig. 1 [16].

The results of careful measurements of isovels (lines of constant axial mean-flow velocity used to indicate velocity variation over the normal duct cross-section) distribution were very surprising – a fact not observed in turbulent flows through straight circular pipes nor in laminar flows through straight rectangular ducts (fig. 2).

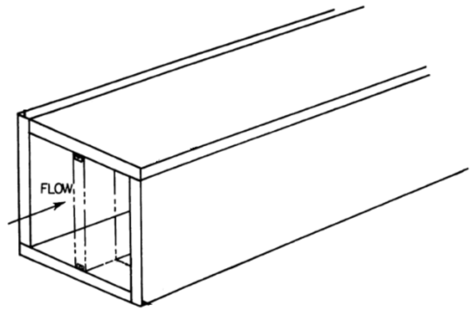

Figure 1. Flow in a straight rectangular duct

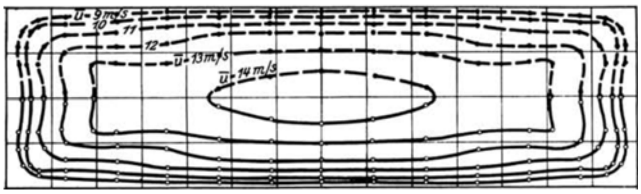

Figure 2. Axial velocity contours (isovels) for a duct of rectangular cross-section [17]

A distribution with even more rounded isovels compared to those obtained in laminar flow was anticipated, but a shape of the axial velocity contours in turbulent flow was unexpectedly different, having convex deformations (increased velocity) in the corner (angle bisectrix) region and concave deformations (decreased velocity) near the wall bisector zone (fig. 3).

Searching for the cause of these peculiar changes, Prandtl recalled an article („Die Wasserkraftlaboratorien Europas“, Berlin, 1926, pp. 66-67, in [17]) on old observations of the spiral motion of water in a straight channel which could be connected with Leonardo da Vinci's famous drawings of water currents in rivers given 500 years ago (fig. 4).

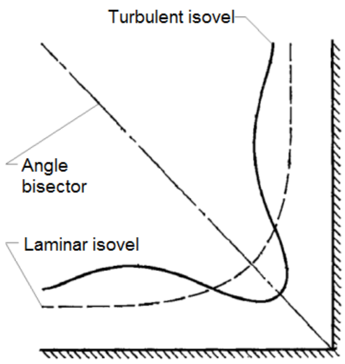

Figure 3. Laminar and turbulent isovels shapes [18, 19]

Figure 4. "Studies of an old man seated and of a swirling water", Windsor, Royal Library, No. 12579, about 1510, Leonardo da Vinci

Using an analogy with curling hair, Leonardo summarized his observations as: Observe the motion of the surface of the water which resembles that of hair, which has two motions, of which one depends on the weight of the hair, the other on the direction of the curls; thus the water forms eddying whirlpools, one part of which is due to the impetus of the principal current, and the other to the incidental motion and return flow (his written comment in fig. 4). It is fascinating how this description, given 500 years ago, is similar to a modern view of the mean flow structure as a superposition of the axial mean flow and transverse mean flow [20]. Namely, Prandtl envisaged that the isovelocity contours distortion was the result of secondary flow – a cross stream circulatory motion superimposed upon the primary axial flow (fig. 5).

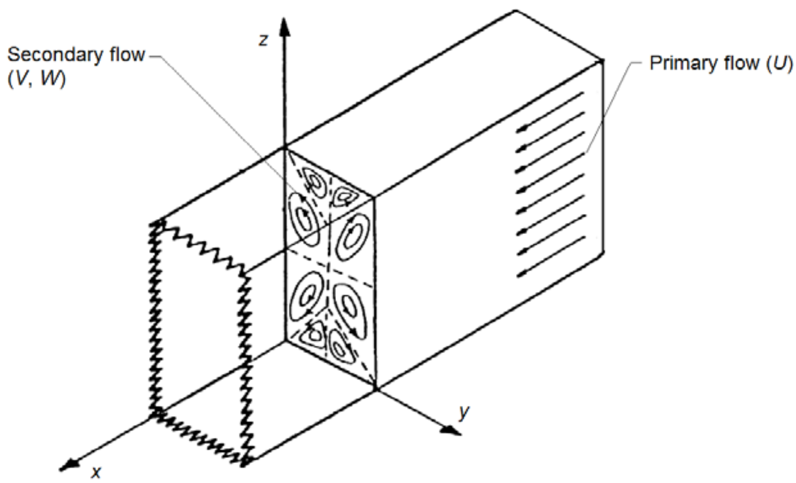

Figure 5. Secondary motions in a straight rectangular duct [21]

Fluid flows from the duct centre to the corner (outward secondary flow) along the bisectrix of the angle (*preferred direction* (geometrical anisotropy) – the longest straight way from the duct centre to the duct wall boundary, square full line, fig. 6), and then toward the duct centre (inward secondary flow) along the wall and along the wall bisector (the shortest way from the duct boundary to the duct centre, square dotted line, fig. 6). In case of the pipe with circular cross-section all straight ways from the pipe center to the pipe wall boundary are of the same length and consequently the *preferred direction* does not exist (geometrical isotropy), fig. 6. The secondary flow circulation mechanism results in a symmetrical system of four pairs of vortical structures, two counter-rotating in each corner in the cross-section, bounded by the walls of the duct and the bisectors of the walls and the corners (opposite pairs – binary opposition). Such paired vortices differ from those usually observed near walls in turbulent boundary layers, because they are large-scale and are locked near the corners by the imposed geometric constraints. By transporting high velocity fluid from the centre of the duct to the corners, relatively high (primary) velocities are generated there (outward secondary flow, isovel convex deformation). In order to satisfy the condition of continuity, by the return flow, the lower momentum fluid is transported into the region of wall bisector creating low velocities there (inward secondary flow, isovel concave deformation), fig. 6. In this way, secondary flow conveys momentum, vorticity, and energy from the centre to the corners, and then, by virtue of continuity, these quantities are transported away from the corner to the centre along the bounding walls and wall bisectors. A similar transport pattern applies for transferable quantities associated with the turbulent motion. These secondary currents have the key role in transport processes between the duct centre and the corners. The mean-flow velocity vector is composed of a component in the axial flow direction (primary velocity, U) and transverse components (secondary velocities, V, W) in a plane normal to this direction. The existence of all three components of the mean velocity vector gives to this kind of flow a 3-D character.

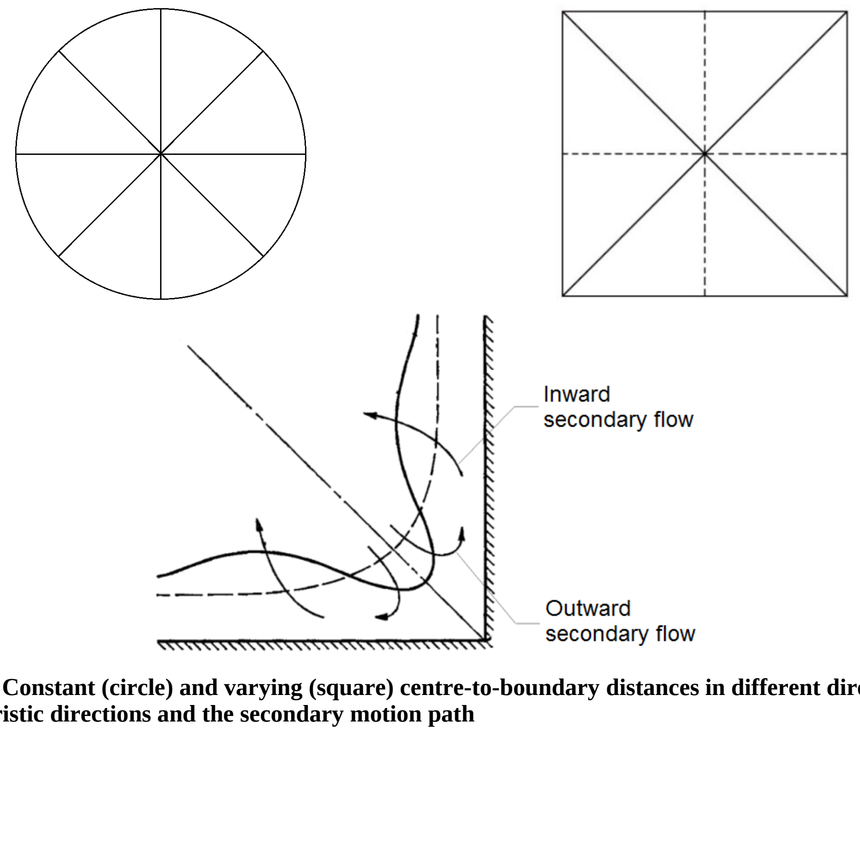

Figure 6. Constant (circle) and varying (square) centre-to-boundary distances in different directions; characteristic directions and the secondary motion path

Shortly afterwards, Nikuradse [22] confirmed the existence of secondary currents by flow visualization studies with dye injection into the water flowing through the duct. Their presence was confirmed in natural streams as well. Using rivers again as an example, Prandtl notes that "we may also mention the fact that small objects floating in rivers tend to move to the middle, which is explained by the existence of a surface current from the banks to the middle".

The first attempt to explain the nature and cause of secondary flows was given by Prandtl [17]. He postulated that turbulent velocity fluctuations tangential to an isovel contour line, in regions where large variations in isovel curvature occurred, resulted in forces proportional to the magnitude of the curvature, causing secondary flow to develop, which was directed from the concave to the convex side of the isovel, towards the corner. It is implied in this assumption that tangential velocity fluctuations (parallel to the isovel) at a given point on a curved isovel are greater than normal velocity fluctuations (perpendicular to the isovel). The later Prandtl's description [22] was based on the variations of wall shear stress along the duct perimeter. The fluid is transported from the center of the duct toward regions with low shear stress, to the corner, where the axial mean velocity is increased, and then into the interior of the channel from regions with high wall shear stress, near the wall bisector, where the axial mean velocity is decreased [10].

Apart from ducts of orthogonal (rectangular) cross-section, Nikuradse investigated non-orthogonal ducts as well, namely a pipe of equilateral triangular cross-section. In subsequent investigations, secondary flows were also encountered in ducts of the square, trapezoidal, rhombic and elliptic cross sections, forming in that way the category of ducts of non-circular cross-section (fig. 7). Typical secondary-flow streamline patterns with symmetrical paired counter-rotating vortical structures initiated a background for the classification of the secondary flows.

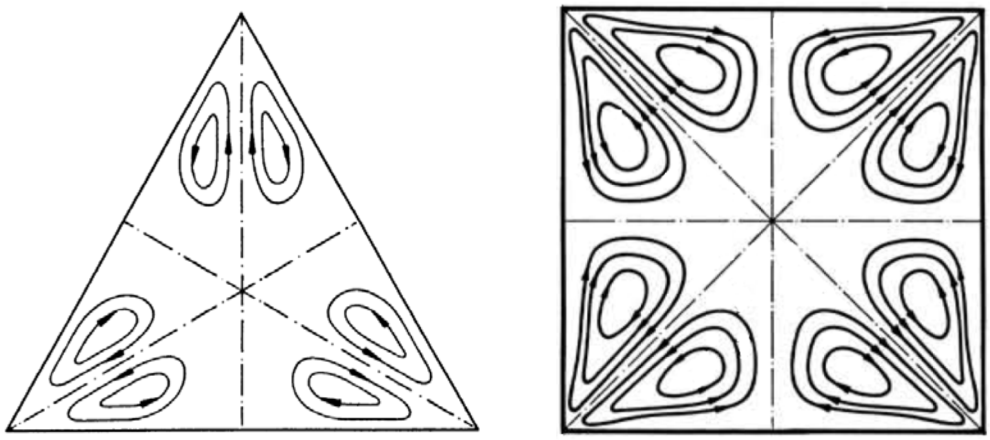

Figure 7. Typical secondary flow streamline patterns in ducts of the triangular and square cross sections [17, 23]

Secondary flows have been formally separated into two classes by Prandtl (fig. 8). The basic division, extended in the course of time, was based on several influential factors: geometrical characteristics of the duct, flow regimes (laminar and turbulent) and physical mechanisms which cause the secondary motion. Secondary flows of the first kind are

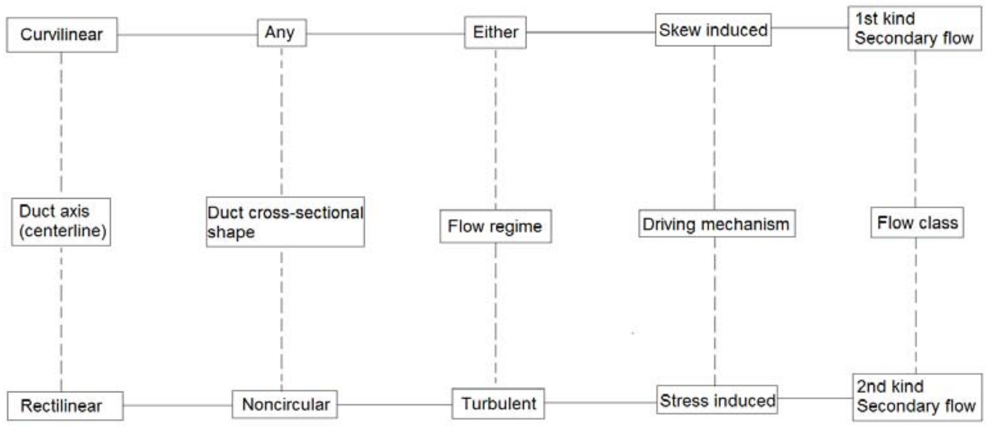

Figure 8. The basic classification of secondary flow phenomena

observed in curved pipes of any cross section. Secondary flows of this type are driven by the centrifugal force and accompanied pressure gradient. Secondary flows of the first kind are also called skew-induced or pressure-driven secondary flows. These flows arise in both laminar and turbulent flow regimes. Secondary flows of the second kind are encountered only in turbulent flows through straight non-circular ducts. Laminar flow in a straight noncircular duct and turbulent flow in a straight circular pipe produce no secondary flow of this type. These flows are turbulence-driven (stress-induced secondary flow) – the gradients of the turbulent Reynolds stresses in the plane of cross-section give rise to a source of streamwise vorticity. Additionally, secondary flow phenomena were found in rotating pipes and ducts (Coriolis force), transition ducts (ducts of non-uniform cross-sectional shape) and in stationary fluid with oscillations (oscillating disks, spheres and cylinders).

Suppose we introduced a term *canonical forms* for the following concepts: a straight line, a circle, and a laminar flow, corresponding to the straight duct (rectilinear duct axis), the pipe of circular cross-section, and the laminar flow through a duct. Using the idea of opposite pairs/binary opposition (straight – curved, circular – non-circular, laminar – turbulent), *non-canonical forms* refer to the curved pipe (curvilinear duct axis), the noncircular cross-section of the duct and the turbulent flow regime. The presence of non-canonical forms leads to the development of secondary flows. Namely, *flow in a curved (the 1st geometrical-static condition of non-canonical form) pipe* is characterized by the formation of secondary motions of Prandtl's first kind. In the case of secondary flows of Prandtl's second kind, since the first condition of the *non-canonical form* is not satisfied (a straight duct), then the following two conditions should be fulfilled, *i. e.* two *non-canonical forms* are needed: the *non-circular* cross-section of the duct (*e. g.* triangular, rectangular, square, rhombic, trapezoidal, elliptic ducts, *etc.*) and the *turbulent* flow regime – *turbulent (the 3rd flow-dynamic condition) flow in a straight non-circular (the 2nd geometrical-static condition) duct*. Laminar flow in a straight noncircular duct and turbulent flow in a straight circular pipe produce no secondary flow. In the case of the secondary flow of the 2nd kind both the geometrical (static) condition and the turbulence (dynamic) condition should be fulfilled, whereas for the secondary flows of the 1st kind only the geometrical (static) condition is needed.

Although detection of secondary flows was comparatively easy, their direct measurement was not. The difficulty lies in the fact that the secondary velocities are, at most, a few percent of the primary velocity. The first actual measurements of the secondary velocities were performed by Hoagland [16]. In spite of the fact that secondary flow is small in magnitude, it has a profound effect on the overall flow (global and local flow properties; mean flow field characteristics as well as turbulence structure). The neglect of the secondary flows tends to produce uniform wall shear stress distribution and heat transfer around the periphery of the duct (analytical results of Deissler [24]), contrary to the experimental measurements.

Vorticity transport equation and driving mechanism of secondary motion

Theoretical analysis of this problem, *i. e.* the description of secondary flow phenomena by equations have been based on: (1) the averaged momentum equation (Reynolds equation), (2) the mean kinetic energy equation, (3) the turbulent kinetic energy equation, and (4) the mean vorticity equation. Chronologically, the first investigations of secondary flow driving mechanism were made from the vorticity point of view. This equation served as a basis for the important experimental studies as well. Secondary flow is closely associated with the existence of a vorticity in the flow field (the fluid elements must possess an angular velocity about an axis in the main flow direction if secondary velocities exist). The vorticity vector has an important role in fluid mechanics as an indicator of rigid-body-like rotation of fluid element. Vorticity is twice the local angular velocity of the fluid. In this paper the symbol ω is adopted for the vorticity vector, although it should be mentioned that this symbol is used for angular velocity vector as well. The vorticity vector is the curl of the velocity vector and can be written in vector or indicial (tensorial) notation by the following expressions:

$$\omega = \nabla \times u \tag{1}$$

$$\omega_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{2}$$

The (instantaneous) vorticity transport equation is derived by applying the curl operator to the Navier-Stokes equations (or by combining successive pairs of the Navier-Stokes momentum equations in a manner so as to eliminate the pressure terms and then simplifying by the use of the continuity equation):

$$\frac{\partial \omega_i}{\partial t} + u_j \frac{\partial \omega_i}{\partial x_j} = \omega_j \frac{\partial u_i}{\partial x_j} + \nu \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} \tag{3}$$

One of the advantages of a description of flow changes in terms of vorticity lies in the absence of the pressure term in eq. (3). On the other hand, the term $\omega_j \partial u_i / \partial x_j$ is the one that has no counterpart in the equation of momentum (Navier-Stokes equation) and that gives vorticity equation a distinctive character.

In turbulent flow, the instantaneous vorticity ω_i is decomposed into a mean vorticity Ω_i and vorticity fluctuations ω_i' ($\omega_i = \Omega_i + \omega_i'$). After substituting this expression and corresponding Reynolds decomposition for instantaneous velocity $u_i = U_i + u_i'$ into eq. (3) and taking the time-average of all terms in the equation, the mean-flow vorticity equation for a steady incompressible constant-property flow is obtained:

$$U_j \frac{\partial \Omega_k}{\partial x_j} = \Omega_j \frac{\partial U_k}{\partial x_j} + \nu \frac{\partial \Omega_k}{\partial x_j \partial x_j} + \varepsilon_{ijk} \frac{\partial^2 (-\overline{u_l^{'} u_j^{'}})}{\partial x_i \partial x_l} \tag{4}$$

As a result of the Reynolds time-averaging procedure, on the right-hand side of the eq. (4) the turbulence Reynolds stress term appeared (the last term).

The presence of secondary flow, described by cross-stream mean velocity components $U_2 = V$ and $U_3 = W$ (fig. 5), implies the existence of mean vorticity component in axial direction, Ω_1, which can be expressed as (from eq. (2)):

$$\Omega_1 = \frac{\partial U_3}{\partial x_2} - \frac{\partial U_2}{\partial x_3} \tag{5}$$

The entire analysis is now focused on the mean streamwise vorticity equation which has the following exact (extended) form (from the compact form of eq. (4)):

$$\begin{aligned} U_1 \frac{\partial \Omega_1}{\partial x_1} + U_2 \frac{\partial \Omega_1}{\partial x_2} + U_3 \frac{\partial \Omega_1}{\partial x_3} &= \Omega_1 \frac{\partial U_1}{\partial x_1} + \Omega_2 \frac{\partial U_1}{\partial x_2} + \Omega_3 \frac{\partial U_1}{\partial x_3} + \nu \left(\frac{\partial^2 \Omega_1}{\partial x_1^2} + \frac{\partial^2 \Omega_1}{\partial x_2^2} + \frac{\partial^2 \Omega_1}{\partial x_3^2} \right) \\ &+ \frac{\partial^2 \left(\overline{u_1^{'} u_2^{'}}\right)}{\partial x_1 \partial x_3} + \frac{\partial^2 \left(\overline{u_2^{'} u_2^{'}}\right)}{\partial x_2 \partial x_3} + \frac{\partial^2 \left(\overline{u_2^{'} u_3^{'}}\right)}{\partial x_3^2} \\ &- \frac{\partial^2 \left(\overline{u_1^{'} u_3^{'}}\right)}{\partial x_1 \partial x_2} - \frac{\partial^2 \left(\overline{u_2^{'} u_3^{'}}\right)}{\partial x_2^2} - \frac{\partial^2 \left(\overline{u_3^{'} u_3^{'}}\right)}{\partial x_2 \partial x_3} \end{aligned} \tag{6}$$

In order to clarify the physical meaning of each term in eq. (6), they are grouped in the following mathematical expressions:

$$A_1 = U_1 \frac{\partial \Omega_1}{\partial x_1} + U_2 \frac{\partial \Omega_1}{\partial x_2} + U_3 \frac{\partial \Omega_1}{\partial x_3} \tag{7}$$

$$A_2 = A_2^{*} + A_2^{**} \tag{8}$$

$$A_2^{*} = \Omega_1 \frac{\partial U_1}{\partial x_1} \tag{9}$$

$$A_2^{**} = \Omega_2 \frac{\partial U_1}{\partial x_2} + \Omega_3 \frac{\partial U_1}{\partial x_3} \tag{10}$$

$$A_3 = \nu \left(\frac{\partial^2 \Omega_1}{\partial x_1^2} + \frac{\partial^2 \Omega_1}{\partial x_2^2} + \frac{\partial^2 \Omega_1}{\partial x_3^2} \right) \tag{11}$$

$$A_4 = \frac{\partial}{\partial x_1}\left(\frac{\partial \overline{u_1' u_2'}}{\partial x_3} - \frac{\partial \overline{u_1' u_3'}}{\partial x_2}\right) \tag{12}$$

$$A_5 = \frac{\partial^2}{\partial x_2 \partial x_3}\left(\overline{u_2'^2} - \overline{u_3'^2}\right) \tag{13}$$

$$A_6 = \left(\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_2^2}\right)\overline{u_2' u_3'} \tag{14}$$

The term A_1 represents the rate of change of mean streamwise vorticity due to convection of fluid by the primary flow (U_1) and secondary motions (U_2, U_3). The term A_2 comprises two physical actions which cause the change of mean axial vorticity ($A_2 = A_2^* + A_2^{**}$). It is suitable to analyze these effects as if vorticity behaved like a material line element coinciding instantaneously with a portion of the vortex-line. Part of the change in vorticity is coming from extension or contraction of the line element (term A_2^*; $j = k$; $U_1 \parallel \Omega_1$ vortex stretching) and part being change in vorticity coming from rigid rotation of line element (term A_2^{**}; $j \neq k$; $U_1 \perp \Omega_2$, $U_1 \perp \Omega_3$; vortex tilting; vortex turning; lateral deflexion of mean flow; skewing of mean flow). The latter effect leads to the exchange of the vorticity between its components, so that y-wise (Ω_2) and z-wise (Ω_3) vortex lines, due to rotation of x-wise vortex line (Ω_1) caused by mean velocity gradients $\partial U_1/\partial x_2$ and $\partial U_1/\partial x_3$, acquire a component in the x-direction (Ω_1). The term A_3 describes the contribution to the rate of change of vorticity due to molecular diffusion of vorticity by viscosity. The terms A_4, A_5, and A_6 express the influence of turbulent Reynolds stresses on the generation of streamwise vorticity.

The physical interpretations of terms in the mean streamwise vorticity equation can be connected with previously mentioned classification of secondary flows into Prandtl's first and second kind. The mean streamwise vorticity in turbulent flow is induced both from meanflow skewing (term A_2^{**}) and from the inhomogeneity of Reynolds stresses (terms A_4, A_5, and A_6). Namely, in curved pipes, the *tilting* or *skewing* mechanism (term A_2^{**}) generates secondary flow of Prandtl's first kind [25]. This mechanism (*skew-induced secondary flow*) can operate either in laminar or in turbulent flow. For turbulent flows in straight non-circular ducts the terms A_4, A_5, and A_6 (gradients of the turbulent Reynolds stresses in the plane of the cross-section) are responsible for developing and maintaining of secondary flow of Prandtl's second kind. This mechanism (*stress-induced or turbulence-driven secondary flow*) cannot induce secondary motions in laminar straight channel flow nor in turbulent straight circular pipe flow. The driving mechanism of stress-induced secondary flows in a straight rectangular duct will be examined in more detail.

For fully developed turbulent flows in straight non-circular ducts, the mean streamwise vorticity equation (eq. 6) can be simplified – all gradients with respect to streamwise direction are zero ($\partial/\partial x_1 = 0$). As a result of this operation, the first addends of the sums in terms A_1 and A_3 vanish, and the term A_4 disappears completely. After some additional mathematical operations, it can be shown that the whole term A_2 reduces to zero as well. The mean streamwise vorticity equation for fully-developed turbulent flow of an incompressible Newtonian fluid in a straight duct of non-circular cross-section then becomes:

$$U_2 \frac{\partial \Omega_1}{\partial x_2} + U_3 \frac{\partial \Omega_1}{\partial x_3} = \nu \left(\frac{\partial^2 \Omega_1}{\partial x_2^2} + \frac{\partial^2 \Omega_1}{\partial x_3^2} \right) + \frac{\partial^2}{\partial x_2 \partial x_3} \left(\overline{u_2'^2} - \overline{u_3'^2} \right) + \left(\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_2^2} \right) \overline{u_2' u_3'} \tag{15}$$

The reduced convection and viscous (diffusion) terms of eq. (15) can be written in the following way:

$$A_1^* = U_2 \frac{\partial \Omega_1}{\partial x_2} + U_3 \frac{\partial \Omega_1}{\partial x_3} \tag{16}$$

$$A_3^* = \nu \left(\frac{\partial^2 \Omega_1}{\partial x_2^2} + \frac{\partial^2 \Omega_1}{\partial x_3^2} \right) \tag{17}$$

The remaining two terms on the right-hand side of eq. (15), Reynolds stresses terms (normal and shear) A_5 and A_6, stay unchanged.

The mathematical form of eq. (15) can be replaced by its physical analogue (from the kinematic point of view), *i. e.* eq. (18), bearing in mind the interconnection between experimental, analytical and modeling/numerical approaches of solving this problem:

$$Convection = Diffusion + Production \tag{18}$$

Since convection by secondary flow serves to transport vorticity from regions of production to regions of diffusion of vorticity by viscosity, where the vorticity is destroyed, it can be concluded that the most influential factor in eq. (18) is the production (generation, source) of vorticity, represented by terms A_5 and A_6. Therefore, the size and distribution of the Reynolds stresses must be considered. In addition, it can be hinted from this equation that axial vorticity can not exist in laminar flows in straight non-circular ducts (*i. e.* secondary flow of the Prandtl's second kind) because the mathematical expression for vorticity production is given by the derivatives of turbulent Reynolds stresses (terms A_5 and A_6).

Examination of the generation mechanism of secondary flows of second kind – the essential information as to where the secondary flow originate and are dissipated – is mainly based on a detailed evaluation of all terms in eq. (15). This equation was investigated both experimentally and theoretically. The earliest recorded measurements of the Reynolds stress tensor components are those of Brundrett and Baines [26] for fully-developed turbulent flow through a straight square duct. The contribution of the term A_6 was found to be negligible, the correlation was an order of magnitude smaller than the term A_5. The subsequent experimental results of Perkins [27] in the corner of a square-sectioned duct suggested that the terms A_5 and A_6 were of equal order, contrary to previous measurements by Brundrett and Baines. The importance of the term A_5 in generating the secondary currents was confirmed and, in addition, the significance of the term A_6 had been demonstrated. The diffusion of vorticity, term A_3^*, is most intense near the wall and towards the corner, whereas it is negligibly small in other duct regions. In later numerical and experimental investigations [28, 29] it was verified that the normal and shear Reynolds stresses terms A_5 and A_6 were the dominant ones, having opposite signs and being much larger than the convection term A_1^*. The difference between A_5 and A_6 is of the same order of magnitude as the convection term A_1^*.This small difference between these relatively large terms is responsible for generating secondary motion. Therefore, both

Reynolds stress terms A_5 and A_6 have to be accurately modeled in order to describe realistically the secondary flow of the second kind.

From a theoretical perspective the important conclusions about the origin of turbulent secondary flow in ducts of noncircular cross-section can be drawn by investigating the conditions under which secondary flow does not exist [21, 23]. In that case there is no axial vorticity, $\Omega_1 = 0$, and secondary velocities are equal to zero as well, $U_2 = U_3 = 0$, so the eq. (15) reduces to:

$$\frac{\partial^2}{\partial x_2 \partial x_3}\left(\overline{u_2'^2} - \overline{u_3'^2}\right) + \left(\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_2^2}\right)\overline{u_2' u_3'} = 0 \tag{19}$$

When the terms of eq. (19) make a non-zero contribution, the streamwise vorticity and consequently secondary flow exist, thus the turbulent Reynolds-stress terms of eq. (19) form vorticity production term or an axial vorticity source term $S_{\Omega 1}$ in eq. (20), *i. e.* it is this term that is the cause of secondary flow. The only exception is fully developed turbulent flow in straight circular pipes where the *ideal symmetry* (geometrical isotropy; a circle has infinitely many lines of symmetry) makes the source term $S_{\Omega 1}$ go to zero:

$$S_{\Omega_1} = \frac{\partial^2}{\partial x_2 \partial x_3}\left(\overline{u_2'^2} - \overline{u_3'^2}\right) + \left(\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_2^2}\right)\overline{u_2' u_3'} \tag{20}$$

The first term on the right-hand side of eq. (20) represents the anisotropy of the turbulent normal stresses, while the second one describes transverse gradients of (secondary) turbulent shear stresses. The complex interaction of these two terms is responsible for the generation of turbulent secondary flows in straight non-circular ducts. This analytical conclusion is in agreement with previously mentioned findings of experimental investigations. In addition, this information can serve as a waymark for identifying the optimal route in a wide variety of turbulence models options.

Secondary flows of Prandtl's second kind and turbulence modeling (RANS)

The experimental and theoretical studies on physical mechanisms that generate and suppress secondary currents in straight ducts of non-circular cross-section significantly contributed to the development of turbulence models for a detailed prediction of this flow configuration. A considerable number of experimental investigations have been carried out to thoroughly understand turbulent flows in straight square/rectangular ducts as one of the simplest geometrical configurations in which turbulence-driven secondary flows arise. Since this relatively simple geometry provides an excellent case to test and develop existing turbulence models, stress-induced secondary flows in straight square/rectangular ducts are of special interest to turbulence modelers.

Despite over a century of research, turbulence remains the major unsolved problem of classical physics. While most researchers agree that the essential physics of turbulent flows can be described by the Navier-Stokes equations, limitations in computer capacity make it impossible – for now and the foreseeable future – to directly solve these equations in the complex turbulent flows of technological interest. Hence, virtually all scientific and engineering calculations of non-trivial turbulent flows, at high Reynolds numbers, are based on some type of modeling. This modeling can take a variety of forms.

Two basic levels of modeling currently used in computational fluid dynamics and transport processes are Eddy Viscosity Models (EVM) and Second-Moment Closure Models (SMC) (known also as Reynolds Stress Models). Each category has a number of variants. The oldest proposal for modeling the turbulent or Reynolds stresses is Boussinesq's eddy-viscosity concept, which assumes that, in analogy to the viscous stresses in laminar flows, the turbulent stresses are proportional to the mean velocity gradient:

$$-\overline{\rho u_i' u_j'} = \mu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3}\rho k \delta_{ij} \tag{21}$$

The main problem in this concept is to find mathematical equations for the eddy viscosity μ_t to model the Reynolds stresses. These may range from the relatively simple algebraic models, to the more complex models such as the k-ε model, where two additional transport equations are solved in addition to the mean flow equations.

As noted in previous section, it is the axial vorticity source term, eq. (20), that leads to the generation of turbulent secondary flows in straight non-circular ducts. According to eq. (21), the axial mean velocity $U_i = (U_1, U_2, U_3) = (U_1, 0, 0) = U$ (fig. 5) produces the following Reynolds stress distribution:

$$\overline{u_2'^2} = \overline{u_3'^2} = \frac{2}{3}k, \quad \overline{u_2' u_3'} = 0 \tag{22}$$

Substituting these values into the eq. (20) we have for the axial vorticity source term:

$$S_{\Omega_1} = 0 \tag{23}$$

Consequently, as a result of eq. (23), it is quite clear that models of turbulence based on the isotropic ($\overline{u_1'^2} = \overline{u_2'^2} = \overline{u_3'^2}$ and $\overline{u_1' u_2'} = \overline{u_1' u_3'} = \overline{u_2' u_3'} = 0$) eddy viscosity concept (*e. g.* k-ε and k-l models) have no built-in mechanism for the creation of secondary flow in straight pipes of noncircular cross section [21].

Since turbulence-driven secondary motions in straight non-circular ducts are an essentially non-isotropic phenomenon, the solution of this problem, from the modeling perspective, has split into two main directions – anisotropic eddy viscosity models and Reynolds stress models. The first path included the correction of deficiency in the k-ε and k-l model by replacing the Boussinesq's linear (isotropic) eddy-viscosity hypothesis with a nonlinear constitutive relationship. Hence, anisotropic eddy viscosity models – by adding non-linear terms to the constitutive relation for the Reynolds stresses – constitute the simplest level of Reynolds stress closure that can predict secondary flows in straight non-circular ducts.

Chronologically, the first attempts were made to model directly the turbulent Reynolds stresses inducing the streamwise vorticity, since the notion of an isotropic turbulent viscosity was inadequate [30]. Most of the subsequent turbulence models developed for predicting turbulent flow in straight non-circular ducts were essentially based on the Reynolds stress models, although they showed considerable variation in the style of approach; examples include equilibrium models, one-equation transport models, two-equation transport models, algebraic stress models and full second-moment closures. All of these models are simplified from the original forms of the Reynolds stress models with various empiric assumptions. Another cause for the perplexity lies in the modeling of near-wall effects – the wall functions (*e. g.* the ambiguity of the wall distance when the corner is approached and the definition of a length scale independent of the wall distance create difficulties) and the near-wall modeling of Reynolds stress models, since both regions close to the wall and the corner are known to influence the characteristics of secondary flow considerably. Therefore, if predictions based on wall functions approach are made, it is difficult to know whether possible problems in both the near-wall and the near-corner regions are due to improperly specified wall functions or to deficiencies in the model itself.

Conclusions

Flow configurations with a dominant flow direction aligned with the duct axis are at the core of traditional engineering calculations. However, the transverse components of the velocity vector or secondary flows in the cross stream plane can develop under certain circumstances (*e. g.* if some kind of *non-canonical forms* exist: 1) flow in a *curved* (the 1st condition of the non-canonical form) pipe or 2) *turbulent* (the 3rd condition) flow in a straight *noncircular* (the 2nd condition) duct. Turbulent flow in straight non-circular ducts, frequently encountered in engineering practice and natural streams, is characterized by the secondary motions in the plane perpendicular to the streamwise direction.

In general, the secondary flow is caused by two different mechanisms and separated into two categories accordingly – secondary flows of Prandtl's first and second kind. Development of secondary flows requires the presence of cross stream gradients in the flow field. On the one hand, cross stream pressure gradients (in balance with centrifugal force) can occur due to turning of the mean flow or rotation of the duct about an axis perpendicular to the mean flow direction. Flow in curved or rotating ducts is characterized by a generation of secondary flows of Prandtl's first kind driven by the centrifugal force and accompanied transverse pressure gradient. It can be observed in both laminar and turbulent flows. On the other hand, transverse gradients can occur due to variations of turbulent stresses over the cross-section in straight non-axisymmetric ducts. Secondary flows of Prandtl's second kind exist in turbulent flows through straight non-circular ducts and cannot arise in circular pipes nor in laminar flows. This flow is turbulence-driven (or stress-induced): it arises from the anisotropy of the transverse turbulence normal stresses and from transverse gradients of the Reynolds shear stresses, both contributing to the generation of streamwise vorticity.

Examination of the generation mechanisms of steady secondary mean flow fields in turbulent flows inside straight non-circular ducts is essentially based on the analysis of the transport equation for the mean streamwise vorticity for fully developed turbulent flow, eq. (15). It was shown, from both the theoretical and experimental standpoints that the axial vorticity source term, eq. (20), led to the generation of turbulent secondary flows in straight non-circular ducts. Furthermore, it was shown that the turbulence models based on the Boussinesq's isotropic hypothesis (k-ε and k-l models) had no natural mechanism for the development of secondary flow. The implications that this had on turbulence modeling were discussed briefly. From a turbulence modeling perspective, there are two possible main routes for prediction of secondary flows: the applying of anisotropic eddy viscosity models or Reynolds stress models. One of the possible obstacles further on the way of solving this problem is the implementation of standard wall functions and their uncertain behavior in the regions close to the wall and the corner which influence the characteristics of secondary flow significantly. It is worth noting that Gessner [19] pointed out equations for two other components of vorticity vector (vorticity of the primary flow, Ω_2 and Ω_3) and the transverse gradients of primary shear stresses ($\overline{u_1'u_2'}$ and $\overline{u_1'u_3'}$) could be even more important for explaining and predicting secondary flows in order to find out whether the mechanism which maintains the secondary flow is equivalent to the mechanism which initiates this transverse flow. This view, however, has not been properly explored yet. Future research is needed to better understand the role that various influential factors play in the development and maintenance of secondary flow, particularly with respect to turbulence modeling close to the wall and corner regions.

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Nomenclature

k – turbulent kinetic energy, [m^{2}s^{-2}]
t – time, [s]
$U, \overline{u}$ – primary (mean axial) velocity, [ms^{-1}]
U_i – mean velocity components, [ms^{-1}]
u_i – instantaneous velocity components, [ms^{-1}]
u_i' – fluctuating velocity components, [ms^{-1}]
$\overline{u_i'u_j'}$ – Reynolds stress components, [m^{2}s^{-2}]
V, W – secondary velocities, [ms^{-1}]
x – Cartesian coordinate, the direction of axial primary flow [m]
x_i – Cartesian coordinates, [m]
y, z – Cartesian coordinates of the duct cross-section, [m]

Greek symbols

δ_{ij} – Kronecker delta, [−]
ε_{ijk} – Levi-Civita symbol (alternating third-order tensor), [−]
μ_t – turbulent or eddy viscosity, [kgm^{-1}s^{-1}]
ν – fluid kinematic viscosity, [m^{2}s^{-1}]
ρ – fluid density, [kgm^{-3}]
Ω_i – mean vorticity components, [s^{-1}]
ω_i – instantaneous vorticity components, [s^{-1}]
ω_i' – fluctuating vorticity components, [s^{-1}]

Subscripts and superscripts

i,j,k,l – coordinate axes indices in tensorial notation (= 1, 2, 3)
t – turbulent
' – fluctuating component

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