LAPLACE TRANSFORM SERIES EXPANSION METHOD FOR SOLVING THE LOCAL FRACTIONAL HEAT-TRANSFER EQUATION DEFINED ON CANTOR SETS

by

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Original scientific paper
DOI: 10.2298/TSCI151217201S

In this paper, we use the Laplace transform series expansion method to find the analytical solution for the local fractional heat-transfer equation defined on Cantor sets via local fractional calculus.

Key words: *heat-transfer equation, analytical solution, local fractional calculus, Laplace transform series expansion method*

Introduction

Heat and mass transfer via local fractional calculus [1] is one of important problems for complex phenomena in applied science. The lots of the local fractional partial differential equations were proposed in fractal heat and mass transfer, such as the heat-conduction [2-4], heat-transfer [5], Laplace and Poisson [6], and oscillator [7] equations.

In this paper, we now consider the local fractional heat-transfer equation defined on Cantor sets [5]:

$$\frac{\partial^{\beta}\Omega(\vartheta,\tau)}{\partial\tau^{\beta}}+\kappa\frac{\partial^{2\beta}\Omega(\vartheta,\tau)}{\partial\vartheta^{2\beta}}+\omega\Omega(\vartheta,\tau)=0 \tag{1}$$

where κ is a heat-diffusive coefficient and ω – a constant related to the density and specific heat of fractal materials.

In eq. (1), the local fractional derivative of $\Omega(\tau)$ is defined [1]:

$$\Omega^{(\beta)}(\tau_0)=\frac{\mathrm{d}^{\beta}\Omega(\tau)}{\mathrm{d}\tau^{\beta}}\Big|_{\tau=\tau_0}=\lim_{\tau\to\tau_0}\frac{\Delta^{\beta}[\Omega(\tau)-\Omega(\tau_0)]}{(\tau-\tau_0)^{\beta}} \tag{2}$$

where $\Delta^{\beta}[\Omega(\tau)-\Omega(\tau_0)]\cong\Gamma(1+\beta)\Delta[\Omega(\tau)-\Omega(\tau_0)]$.

The inverse operator of eq. (2) is defined [1]:

$$\Omega(\tau)={}_aI_b^{(\beta)}\theta(\tau)=\frac{1}{\Gamma(1+\beta)}\int_a^b\theta(\tau)(\mathrm{d}\tau)^{\beta}=\frac{1}{\Gamma(1+\beta)}\lim_{\Delta\tau\to 0}\sum_{j=0}^{j=N-1}\theta(\tau)(\Delta\tau)^{\beta} \tag{3}$$

where $\Delta\tau=\tau_{j+1}-\tau_j$, $j=0,...,N-1$, $\tau_0=a$, $\tau_N=b$.

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The local fractional Laplace transform (LFLT) of $\Omega(\tau)$ is defined [1]:

$$\tilde{Y}_\beta\{\Omega(\tau)\} = \Omega_y^{\tilde{Y},\beta}(y) = \frac{1}{\Gamma(1+\beta)}\int_0^\infty E_\beta(-y^\beta\tau^\beta)\Omega(\tau)(\mathrm{d}\tau)^\beta, \qquad 0<\beta\le 1 \tag{4}$$

The inverse LFLT is defined [1]:

$$\Omega(\tau) = \tilde{Y}_\beta^{-1}\{\Omega_y^{\tilde{Y},\beta}(y)\} = \frac{1}{(2\pi)^\beta}\int_{\mu-i\infty}^{\mu+i\infty} E_\beta(y^\beta\tau^\beta)\Omega_y^{\tilde{Y},\beta}(y)(\mathrm{d}y)^\beta \tag{5}$$

where $y^\beta = \mu^\beta + i^\beta\infty^\beta$ and $\mathrm{Re}(y^\beta) = \mu^\beta$.

The some properties of the LFLT are given [1]:

$$\tilde{Y}_\beta\{a\Omega_1(\tau) + b\Omega_2(\tau)\} = a\tilde{Y}_\beta\{\Omega_1(\tau)\} + b\tilde{Y}_\beta\{\Omega_2(\tau)\} \tag{6}$$

$$\tilde{Y}_\beta\{\Omega^{(n\beta)}(\tau)\} = y^{n\beta}\tilde{Y}_\beta\{\Omega(\tau)\} - \sum_{k=1}^{n} y^{(k-1)\beta}\Omega^{(n-k)\beta}(0) \tag{7}$$

$$\tilde{Y}_\beta\{E_\beta(\tau^\beta)\} = \frac{1}{y^\beta - 1} \tag{8}$$

$$\tilde{Y}_\beta\left\{\frac{\tau^{k\beta}}{\Gamma(1+k\beta)}\right\} = \frac{1}{y^{(k+1)\beta}} \tag{9}$$

Based on the LFLT, Yan [8] proposed the local fractional Laplace series expansion method. The goal of the paper is to use the local fractional Laplace series expansion method to solve the local fractional heat-transfer equation defined on Cantor sets.

Solving the local fractional heat-transfer equation defined on Cantor sets

Following the idea of the local fractional Laplace series expansion method [8], we write eq. (1) in the form:

$$\Omega_\tau^{(\beta)} = \Pi_\beta\Omega \tag{10}$$

where

$$\Omega_\tau^{(\beta)} = \frac{\partial^\beta\Omega(\vartheta,\tau)}{\mathrm{d}\tau^\beta} \qquad \text{and} \qquad \Pi_\beta = -\left(\kappa\frac{\partial^{2\beta}}{\partial\vartheta^{2\beta}} + \omega\right)$$

is a linear local operator with respect to ϑ.

Let us consider:

$$\Omega(\vartheta,\tau) = \sum_{i=0}^{\infty}\Lambda_i(\tau)\varpi_i(\vartheta) \tag{11}$$

where $\Lambda_i(\tau)$ and $\varpi_i(\vartheta)$ are two functions defined on Cantor sets.

By assuming $\Lambda_i(\tau)=\tau^{i\beta}/\Gamma(1+i\beta)$, eq. (11) becomes:

$$\Omega(\vartheta,\tau)=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_i(\vartheta) \tag{12}$$

Taking the LFLT of eq. (11) gives:

$$\Omega(y,\tau)=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_i(y) \tag{13}$$

Thus, we have:

$$\tilde{Y}_\beta\{\Omega_\tau^{(\beta)}(\tau,\vartheta)\}=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_{i+1}(y)=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_{i+1}(y) \tag{14}$$

$$\tilde{Y}_\beta\{\Pi_\beta\Omega(\tau,\vartheta)\}=\Pi_\beta\left[\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_i(y)\right]=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}(\Pi_\beta\varpi_i)(y) \tag{15}$$

From eqs. (14) and (15) we have:

$$\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_{i+1}(y)=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}(\Pi_\beta\varpi_i)(y) \tag{16}$$

which reduces to:

$$\varpi_{i+1}(y)=(\Pi_\beta\varpi_i)(y) \tag{17}$$

In view of eq. (16), we have:

$$\Omega(y,\tau)=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_i(y) \tag{18}$$

such that:

$$\Omega(\vartheta,\tau)=\tilde{Y}_\beta^{-1}\{\Omega(y,\tau)\}=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\tilde{Y}_\beta^{-1}\{\varpi_i(y)\} \tag{19}$$

Let us consider the initial condition:

$$\Omega(\vartheta,0)=E_\beta(\vartheta^\beta) \tag{20}$$

We have the following iterative formula:

$$\begin{cases}\varpi_{i+1}(y)=(\Pi_\beta\varpi_i)(y)=-(\kappa+\omega)\varpi_i(y)\\ \varpi_0(y)=\dfrac{1}{y^\beta-1}\end{cases} \tag{21}$$

Thus, we have:

$$\varpi_1(y)=\frac{-(\kappa+\omega)}{y^\beta-1}, \qquad \varpi_2(y)=\frac{[-(\kappa+\omega)]^2}{y^\beta-1}, \qquad \ldots, \qquad \varpi_i(y)=\frac{[-(\kappa+\omega)]^i}{y^\beta-1} \tag{22}$$

Therefore, we obtain the series:

$$\Omega(y,\tau)=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\varpi_i(y)=\sum_{i=0}^{\infty}\frac{\tau^{i\beta}}{\Gamma(1+i\beta)}\frac{[-(\kappa+\omega)]^i}{y^\beta-1} \tag{23}$$

With the help of eqs. (8) and (23), we get the solution of eq. (1):

$$\Omega(\vartheta,\tau)=\sum_{i=0}^{\infty}\frac{\left[-(\kappa+\omega)\right]^i\tau^{i\beta}}{\Gamma(1+i\beta)}E_\beta(\vartheta^\beta) \tag{24}$$

Conclusion

In this paper, we considered the local fractional heat-transfer equation defined on Cantor sets. The local fractional Laplace series expansion method was used to find its solution of non-differentiable type. The obtained result shows the efficiency and accuracy of the proposed technology for the fractal heat-transfer problem via local fractional calculus.

Nomenclature

ϑ – space co-ordinate, [m]
$\Omega(\vartheta,\tau)$ – concentration, [K]
β – fractal dimensional order, [–]
τ – time, [s]
$\tilde{Y}_\beta\{\Omega(\tau)\}$ – LFLT of $\Omega(\tau)$, [–]
$\tilde{Y}_\beta^{-1}\{\Omega_y^{\tilde{Y},\beta}(y)\}$ – inverse LFLT of $\Omega_y^{\tilde{Y},\beta}(y)$, [–]

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Paper submitted: December 17, 2015
Paper revised: January 14, 2016
Paper accepted: January 26, 2016