TIME-FRACTIONAL FREE CONVECTION FLOW NEAR A VERTICAL PLATE WITH NEWTONIAN HEATING AND MASS DIFFUSION

by

Dumitru VIERU[a*], Constantin FETECAU[b,c], and Corina FETECAU[a]

[a] Department of Theoretical Mechanics, Technical University of Iasi, Iasi, Romania
[b] Department of Mathematics, Technical University of Iasi, Iasi, Romania
[c] Academy of Romanian Scientists, Bucuresti, Romania

Original scientific paper
DOI: 10.2298/TSCI15S1S85V

The time-fractional free convection flow of an incompressible viscous fluid near a vertical plate with Newtonian heating and mass diffusion is investigated in presence of first order chemical reaction. The dimensionless temperature, concentration, and velocity fields, as well as the skin friction and the rates of heat and mass transfer from the plate to the fluid, are determined using the Laplace transform technique. Closed form expressions are established in terms of Robotnov-Hartley and Wright functions. The similar solutions for ordinary fluids are also determined. Finally, the influence of fractional parameter on the temperature, concentration and velocity fields is graphically underlined and discussed.

Key words: *time-fractional free convection flow, Newtonian heating, mass diffusion, chemical reaction*

Introduction

Free or natural convection flow of an incompressible viscous fluid near a vertical plate was extensively studied due to its vast industrial applications [1, 2]. Since 1971, Gebhart and Pera [3] have studied the vertical natural convection flow resulting from the combined buoyancy effects of thermal and mass diffusion. Later, Chamkha *et al*. [4] and Ganesan and Loganathan [5] studied the radiation effects on the free convection flow with mass transfer near a semi infinite vertical plate, respectively, past a moving cylinder. Over time, different publications of this type appeared but an increasing interest has been evinced in the radiation interaction with convection and chemical reaction. In many engineering processes the chemical reactions play an important role in heat and mass transfer. A chemical reaction is said to be of the first order if its rate of reaction is directly proportional to the concentration [6]. Many researchers have studied the effects of chemical reaction under different conditions on the convective flow with heat and mass transfer. Some of the most recent and interesting studies of this kind are those of Mahapatra *et al*. [7], Sharma *et al*. [8], Muthucumaraswamy and Shankar [9], Reddy *et al*. [10, 11], Ahmed and Dutta [12], Reddy *et al*. [13], and Srihari *et al*. [14].

However, in all these studies the flow is driven by a prescribed surface temperature or by a surface heat flux. In this work we assume that the flow is set up by Newtonian heating, *i. e*. the heat transfer from the surface is proportional to the local surface temperature. The Newtonian heating, with important applications in engineering, was initiated by Merkin [15] and its effects on the free convection flow over an infinite plate have been investigated by

* Corresponding author; e-mail: dumitru_vieru@yahoo.com

many authors. A few of the most recent and important results in this field seem to be those of Narahari *et al.* [16], Narahari and Dutta [17], Ramzan *et al.* [18], Hussanan *et al.* [19, 20], and Vieru *et al.* [21].

In the last time, the fractional calculus has been extensively used to describe the viscoelastic behavior of materials. It is increasingly seen as an efficient tool through which useful generalizations of physical concepts can be obtained [22]. Usually, the governing equations for fractional fluids are obtained from those of ordinary fluids by replacing time derivatives of an integer order with fractional derivatives of order α. In the case of diffusion phenomena [23], for instance, $\alpha = 1$ corresponds to the classical diffusion while for $0 < \alpha < 1$ or $\alpha > 1$ the transport phenomenon exhibits sub-diffusion, respectively, super-diffusion. To the best of our knowledge, the fractional calculus has not been used in convection problems with Newtonian heating and mass transfer.

The purpose of this work is to study the time-fractional free convection flow of an incompressible viscous fluid near a vertical plate with Newtonian heating and chemical reaction. The radiative effects are not taken into consideration but, according to Magyari and Pantokratoras [24], they can be easy included by a simple re-scaling of the Prandtl number. The Laplace transform technique is used to determine closed-form expressions for velocity, temperature, concentration, skin friction and the rates of heat and mass transfer from the plate to the fluid. The solutions corresponding to ordinary fluids are also determined. In the absence of chemical reaction, as expected, the expressions of temperature and concentration reduce to the corresponding solutions of Narahari and Dutta [17]. Finally the influence of fractional parameter on the temperature, concentration and velocity is graphically underlined and discussed.

Mathematical formulation

The governing equations corresponding to the unsteady free convection flow of an incompressible viscous fluid over an infinite vertical plate with mass diffusion and chemical reaction, as it results from [12] and [17], are given by:

$$\frac{\partial u(y,t)}{\partial t} = \nu \frac{\partial^2 u(y,t)}{\partial y^2} + \mathrm{g}\beta\left[T(y,t) - T_\infty\right] + \mathrm{g}\gamma\left[C(y,t) - C_\infty\right] \tag{1}$$

$$\rho C_p \frac{\partial T(y,t)}{\partial t} = k \frac{\partial^2 T(y,t)}{\partial y^2} \tag{2}$$

$$\frac{\partial C(y,t)}{\partial t} = D \frac{\partial^2 C(y,t)}{\partial y^2} - K\left[C(y,t) - C_\infty\right] \tag{3}$$

Equations (1)-(3) are obtained under the usual Boussinesq approximation [17, 24] when the viscous dissipation of energy is negligible. Furthermore, as it was shown in [24], the effects of thermal radiation in the linearized Rosseland approximation are quite trivial both physically and computationally and they will be finally included by a simple re-scaling of the Prandtl number with a factor involving the radiation parameter. The appropriate initial and boundary conditions are:

$$u(y,0) = 0, \quad T(y,0) = T_\infty, \quad C(y,0) = C_\infty; \quad y \ge 0 \tag{4}$$

$$u(0,t) = 0, \quad \left.\frac{\partial T(y,t)}{\partial y}\right|_{y=0} = -\frac{h}{k} T(0,t), \quad C(0,t) = C_w; \quad t > 0 \tag{5}$$

$$u(y,t)\to 0,\quad T(y,t)\to T_\infty,\quad C(y,t)\to C_\infty \quad \text{as} \quad y\to\infty \tag{6}$$

In order to develop a model with fractional derivatives, we firstly multiply eqs. (1)-(3) by $\lambda = \nu h/gk$ and then replace λ and the partial derivatives with respect to t from the left parts of the obtained equations by λ^α, respectively, D_t^α where:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(s)}{(t-s)^\alpha}\mathrm{d}s \quad \text{if} \quad 0<\alpha<1; \quad D_t^\alpha f(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} \quad \text{if} \quad \alpha = 1 \tag{7}$$

is the Caputo fractional differential operator of order α. A simple analysis clearly shows that λ has the dimension of the time t. In order to determine solutions that are independent of the geometry of flow regime we also introduce the following dimensionless variables and parameters:

$$y^* = \frac{h}{k}y,\quad t^* = \frac{t}{\lambda},\quad u^* = \frac{k}{\nu h}u,\quad T^* = \frac{T-T_\infty}{T_\infty},\quad C^* = \frac{C-C_\infty}{C_w - C_\infty},\quad \mathrm{Pr} = \frac{\mu C_p}{k},$$

$$\mathrm{Re} = \frac{\nu^2}{g}\left(\frac{h}{k}\right)^3,\quad \mathrm{Gr} = \beta T_\infty,\quad \mathrm{Gm} = \gamma(C_w - C_\infty),\quad \mathrm{Sc} = \frac{\nu}{D},\quad K^* = \frac{\nu h}{\mathrm{g}k}K \tag{8}$$

The dimensionless forms of the governing equations, dropping out the * notation, are:

$$D_t^\alpha u(y,t) = \mathrm{Re}\frac{\partial^2 u(y,t)}{\partial y^2} + \mathrm{Gr}\,T(y,t) + \mathrm{Gm}\,C(y,t);\quad y,\quad t>0 \tag{9}$$

$$\mathrm{Pr}_{eff}\, D_t^\alpha T(y,t) = \frac{\partial^2 T(y,t)}{\partial y^2};\quad y,\quad t>0 \tag{10}$$

$$\mathrm{Sc}_{\mathrm{eff}}\, D_t^\alpha C(y,t) = \frac{\partial^2 C(y,t)}{\partial y^2} - K\,\mathrm{Sc}_{\mathrm{eff}}\, C(y,t);\quad y,\quad t>0 \tag{11}$$

The notion of effective Prandtl number $\mathrm{Pr}_{\mathrm{eff}}$, but with a little different signification, has been firstly introduced by Magyari and Pantokratoras [24] showing that a two parameter approach is superfluous.

The corresponding initial and boundary conditions are:

$$u(y,0) = 0,\quad T(y,0) = 0,\quad C(y,0) = 0;\quad y\ge 0 \tag{12}$$

$$u(0,t) = 0,\quad \left.\frac{\partial T(y,t)}{\partial y}\right|_{y=0} = -\left[T(0,t)+1\right],\quad C(0,t) = 1;\quad t>0 \tag{13}$$

$$u(y,t)\to 0,\quad T(y,t)\to 0,\quad C(y,t)\to 0 \quad \text{as} \quad y\to\infty,\quad t\ge 0 \tag{14}$$

Solution of the problem

The partial differential eqs. (10) and (11) are not coupled to the momentum eq. (9). Consequently, we shall firstly determine the temperature and concentration fields by means of the Laplace transform technique and then the fluid velocity.

Calculation of the temperature field

Applying the Laplace transform with respect to the temporal variable t to eq. (10) and using the corresponding initial and boundary conditions, we find that:

$$\mathrm{Pr_{eff}}\, q^{\alpha}\bar{T}(y,q)=\frac{\partial^2\bar{T}(y,q)}{\partial y^2};\quad y>0 \tag{15}$$

where q is the transform parameter and the Laplace transform $\bar{T}(y,q)$ of $T(y,t)$ has to satisfy the conditions:

$$\left.\frac{\partial\bar{T}(y,q)}{\partial y}\right|_{y=0}=-\left[\bar{T}(0,q)+\frac{1}{q}\right]\quad\text{and}\quad \bar{T}(y,q)\to 0\quad\text{as}\quad y\to\infty \tag{16}$$

The solution of the ordinary differential eq. (15) with the boundary conditions (16) can be written under the suitable form:

$$\bar{T}(y,q)=\frac{1}{\sqrt{\mathrm{Pr_{eff}}\, q^{\alpha}}-1}\frac{\mathrm{e}^{-y\sqrt{\mathrm{Pr_{eff}}\, q^{\alpha}}}}{q} \tag{17}$$

Now, applying the inverse Laplace transform to eq. (17) and using eqs. (A1) and (A2) from Appendix, as well as the convolution theorem, we find that:

$$T(y,t)=\frac{1}{\sqrt{\mathrm{Pr_{eff}}}}\int_0^t F_{\alpha/2}\left(\frac{1}{\sqrt{\mathrm{Pr_{eff}}}},t-s\right)\Phi\left(1,-\frac{\alpha}{2};\quad -y\sqrt{\mathrm{Pr_{eff}}}\,s^{-\alpha/2}\right)\mathrm{d}s \tag{18}$$

where $\mathrm{F}_{\mu}(a,t)$ is the F-function of Robotnov and Hartley [25] and $\Phi(a,b;c)$ is the Wright function [26].

In the special case when $\alpha=1$, we recover the solution:

$$T(y,t)=\exp\left(-y+\frac{t}{\mathrm{Pr_{eff}}}\right)\mathrm{erfc}\left(\frac{y\sqrt{\mathrm{Pr_{eff}}}}{2\sqrt{t}}-\sqrt{\frac{t}{\mathrm{Pr_{eff}}}}\right)-\mathrm{erfc}\frac{y\sqrt{\mathrm{Pr_{eff}}}}{2\sqrt{t}} \tag{19}$$

obtained by Narahari and Dutta [17, eq. (14)].

The local coefficient of the rate of heat transfer from the plate to the fluid, in terms of Nusselt number, namely:

$$\mathrm{Nu}=1+\frac{1}{\sqrt{\mathrm{Pr_{eff}}}}t^{\alpha/2}E_{\frac{\alpha}{2},\frac{\alpha}{2}+1}\left(\frac{1}{\sqrt{\mathrm{Pr_{eff}}}}t^{\alpha/2}\right)\quad\text{or}\quad \mathrm{Nu}=\exp\left(\frac{t}{\mathrm{Pr_{eff}}}\right)\left(2-\mathrm{erfc}\sqrt{\frac{t}{\mathrm{Pr_{eff}}}}\right) \tag{20}$$

for $\alpha\in(0,1)$, respectively, $\alpha=1$ is obtained by introducing $\bar{T}(y,q)$ into relation (see [12, eq. (45)]):

$$\mathrm{Nu}=-\left.\frac{\partial T(y,t)}{\partial y}\right|_{y=0}=-\frac{\partial}{\partial y}L^{-1}[\bar{T}(y,q)]\Big|_{y=0}=-L^{-1}\left[\left.\frac{\partial\bar{T}(y,q)}{\partial y}\right|_{y=0}\right] \tag{21}$$

and using eq. (A3) from the Appendix. Here, $E_{a,b}(z)$ is the well-known Mittag-Leffler function.

Calculation of the concentration field

Applying the Laplace transform to eq. (11) and using the corresponding initial and boundary conditions, it results that:

$$(q^{\alpha}+K)\bar{C}(y,q)=\frac{1}{\mathrm{Sc_{eff}}}\frac{\partial^2\bar{C}(y,q)}{\partial y^2};\quad y>0 \tag{22}$$

where the Laplace transform $\bar{C}(y,q)$ of $C(y,t)$ has to satisfy the conditions:

$$\bar{C}(0,q)=\frac{1}{q}\quad\text{and}\quad \bar{C}(y,q)\to 0\quad\text{as}\quad y\to\infty \tag{23}$$

Now, in order to determine the expression of $C(y,t)$, we write the solution of eq. (22) with the boundary conditions (23) under the suitable form:

$$\bar{C}(y,q)=\frac{q^{\alpha}+K}{q}\frac{\exp\left[-y\sqrt{\mathrm{Sc_{eff}}\,(q^{\alpha}+K)}\right]}{q^{\alpha}+K} \tag{24}$$

Applying the inverse Laplace transform to eq. (24) and using again the convolution theorem as well as eqs. (A2), (A4), and the property (A5) from the Appendix, we find that:

$$C(y,t)=\int_0^{\infty}\mathrm{e}^{-Ku}\mathrm{erfc}\frac{y\sqrt{\mathrm{Sc_{eff}}}}{2\sqrt{u}}\int_0^t\frac{1}{s}\left[K+\frac{1}{\Gamma(1-\alpha)(t-s)^{\alpha}}\right]\Phi(0,-\alpha;-us^{-\alpha})\,\mathrm{d}s\mathrm{d}u,\quad \text{if}\ \ \alpha\in(0,1) \tag{25}$$

In the case of $\alpha=1$, the corresponding solution, see [27]:

$$C(y,t)=\frac{1}{2}\left[\mathrm{e}^{-y\sqrt{K\,\mathrm{Sc_{eff}}}}\,\mathrm{erfc}\left(\frac{y\sqrt{\mathrm{Sc_{eff}}}}{2\sqrt{t}}-\sqrt{Kt}\right)+\mathrm{e}^{y\sqrt{K\,\mathrm{Sc_{eff}}}}\,\mathrm{erfc}\left(\frac{y\sqrt{\mathrm{Sc_{eff}}}}{2\sqrt{t}}+\sqrt{Kt}\right)\right] \tag{26}$$

as expected, reduces as form to eq. (13a) from [17], when $K=0$.

The local coefficient of the rate of mass transfer from the plate to the fluid, in terms of the Sherwood number, namely:

$$\mathrm{Sh}=\sqrt{\mathrm{Sc_{eff}}}\left[G_{\alpha,\alpha\text{-}1,1/2}(-K,t)+KG_{\alpha,\text{-}1,1/2}(-K,t)\right]$$

or

$$\mathrm{Sh}=\sqrt{\mathrm{Sc_{eff}}}\left[\frac{\mathrm{e}^{-Kt}}{\sqrt{\pi t}}+\sqrt{K}\,\mathrm{erf}(\sqrt{Kt})\right] \tag{27}$$

for $\alpha\in(0,1)$, respectively, $\alpha=1$ is obtained introducing $\bar{C}(y,q)$ into relation [12, eq. (47)]:

$$\mathrm{Sh}=-\frac{\partial C(y,t)}{\partial y}\bigg|_{y=0}=-\frac{\partial}{\partial y}L^{-1}\left[\bar{C}(y,q)\right]\bigg|_{y=0}=-L^{-1}\left\{\frac{\partial\bar{C}(y,q)}{\partial y}\bigg|_{y=0}\right\} \tag{28}$$

and using eq. (A6) from the Appendix where $G_{a,b,c}(d,t)$ is the G-function of Lorenzo-Hartley [28].

Calculation of velocity

Applying the Laplace transform to eq. (9) and bearing in mind the associated initial and boundary conditions as well as the previous expressions of $\bar{T}(y,q)$ and $\bar{C}(y,q)$, it results that:

$$q^{\alpha}\bar{u}(y,q)=\mathrm{Re}\frac{\partial^2\bar{u}(y,q)}{\partial y^2}+\frac{\mathrm{Gr}}{q\left(\sqrt{\mathrm{Pr}_{\mathrm{eff}}\,q^{\alpha}}-1\right)}\exp\left(-y\sqrt{\mathrm{Pr}_{\mathrm{eff}}\,q^{\alpha}}\right)+$$

$$+\frac{\mathrm{Gm}}{q}\exp\left[-y\sqrt{\mathrm{Sc}_{\mathrm{eff}}\,(q^{\alpha}+K)}\right] \tag{29}$$

where the Laplace transform $\bar{u}(y,q)$ of $u(y,t)$ has to satisfy the conditions:

$$\bar{u}(0,q)=0 \quad \text{and} \quad \bar{u}(y,q)\to 0 \quad \text{as} \quad y\to\infty \tag{30}$$

The solution of eq. (29) subject to the conditions (30) is:

$$\bar{u}(y,q)=\frac{\mathrm{Gr}\sqrt{\mathrm{Re}}}{1-\mathrm{Pr}}\frac{1}{q^{\alpha+1}\left(\sqrt{\mathrm{Pr}\,q^{\alpha}}-\sqrt{\mathrm{Re}}\right)}\left[\exp\left(-y\sqrt{\frac{\mathrm{Pr}}{\mathrm{Re}}q^{\alpha}}\right)-\exp\left(-y\sqrt{\frac{q^{\alpha}}{\mathrm{Re}}}\right)\right]+$$

$$+\frac{\mathrm{Gm}}{q[(1-\mathrm{Sc})q^{\alpha}-K\,\mathrm{Sc}]}\left[\exp\left(-y\sqrt{\frac{\mathrm{Sc}}{\mathrm{Re}}(q^{\alpha}+K)}\right)-\exp\left(-y\sqrt{\frac{q^{\alpha}}{\mathrm{Re}}}\right)\right] \tag{31}$$

Finally, in order to obtain the (y,t)-domain solution for velocity, namely:

$$u(y,t)=\frac{\mathrm{Gr}\sqrt{\mathrm{Re}}}{(1-\mathrm{Pr})\sqrt{\mathrm{Pr}}}\int_0^t \mathrm{F}_{\alpha/2}\left(\sqrt{\frac{\mathrm{Re}}{\mathrm{Pr}}},t-s\right)\left[\Phi\left(\alpha+1,-\frac{\alpha}{2};-y\sqrt{\frac{\mathrm{Pr}}{\mathrm{Re}\,s^{\alpha}}}\right)-\right.$$

$$\left.-\Phi\left(\alpha+1,-\frac{\alpha}{2};-\frac{y}{\sqrt{\mathrm{Re}\,s^{\alpha}}}\right)\right]s^{\alpha}\mathrm{d}s+$$

$$+\frac{\mathrm{Gm}}{1-\mathrm{Sc}}\int_0^{\infty}\mathrm{e}^{-Ku}\mathrm{erfc}\frac{y\sqrt{\mathrm{Sc}}}{2\sqrt{\mathrm{Re}\;u}}\int_0^t\frac{1}{s}\Phi(0,-\alpha;-us^{-\alpha})\,\mathrm{d}s\mathrm{d}u-$$

$$-\frac{\mathrm{Gm}}{1-\mathrm{Sc}}\int_0^t F_{\alpha}\left(\frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}},t-s\right)\Phi\left(1,-\frac{\alpha}{2};-\frac{y}{\sqrt{\mathrm{Re}\,s^{\alpha}}}\right)\mathrm{d}s+$$

$$+\frac{\mathrm{Gm}\,K}{(1-\mathrm{Sc})^2}\int_0^{\infty}\mathrm{e}^{-Ku}\mathrm{erfc}\frac{y\sqrt{\mathrm{Sc}}}{2\sqrt{\mathrm{Re}u}}\int_0^t\frac{(t-s)^{\alpha}}{s}\mathrm{E}_{\alpha,\alpha+1}\left[\frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}}(t-s)^{\alpha}\right]\Phi(0,-\alpha;-us^{-\alpha})\mathrm{d}s\mathrm{d}u \tag{32}$$

we apply the inverse Laplace transform to eq. (31) and use eqs. (A1)-(A3) and (A5) from the Appendix.

The solution corresponding to $\alpha=1$, namely:

$$u(y,t)=\frac{\mathrm{Gr}\sqrt{\mathrm{Re}}}{(1-\mathrm{Pr})\sqrt{\mathrm{Pr}}}\left[\varphi\left(y\sqrt{\frac{\mathrm{Pr}}{\mathrm{Re}}},\ -\sqrt{\frac{\mathrm{Re}}{\mathrm{Pr}}},\ t\right)-\varphi\left(\frac{y}{\sqrt{\mathrm{Re}}},\ -\sqrt{\frac{\mathrm{Re}}{\mathrm{Pr}}},\ t\right)\right]-$$

$$-\frac{\mathrm{Gm}}{K\,\mathrm{Sc}}\left[\Psi\left(y\sqrt{\frac{\mathrm{Sc}}{\mathrm{Re}}},\ K,\ 0,\ t\right)-\Psi\left(y\sqrt{\frac{\mathrm{Sc}}{\mathrm{Re}}},\ K,\ \frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}},\ t\right)+\right.$$

$$\left.+\Psi\left(\frac{y}{\sqrt{\mathrm{Re}}},\ 0,\ \frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}},\ t\right)-\Psi\left(\frac{y}{\sqrt{\mathrm{Re}}},\ 0,\ 0,\ t\right)\right] \tag{33}$$

is obtained by means of the eqs. (A7) and (A8) from the Appendix.

The skin friction in non-dimensional form is:

$$\tau=-\left.\frac{\partial u(y,t)}{\partial y}\right|_{y=0}=-\frac{\partial}{\partial y}L^{-1}\left[\bar{u}(y,q)\right]\Big|_{y=0}=-L^{-1}\left\{\left.\frac{\partial\bar{u}(y,q)}{\partial y}\right|_{y=0}\right\} \tag{34}$$

Introducing $\bar{\mathrm{u}}(y,q)$ from eq. (31) into (34) and using eqs. (A1) and (A6), we find that:

$$\tau=-\frac{\mathrm{Gr}}{\sqrt{\mathrm{Pr}}(\sqrt{\mathrm{Pr}}+1)}\mathrm{G}_{\frac{\alpha}{2},-\frac{\alpha}{2}-1,1}\left(\sqrt{\frac{\mathrm{Re}}{\mathrm{Pr}}},\ t\right)-\frac{\mathrm{Gm}}{\sqrt{\mathrm{Re}}(1-\mathrm{Sc})}\mathrm{G}_{\alpha,\frac{\alpha}{2}-1,1}\left(\frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}},\ t\right)+$$

$$+\frac{\mathrm{Gm}\sqrt{\mathrm{Sc}}}{\sqrt{\mathrm{Re}}(1-\mathrm{Sc})}\int_0^t\left[\mathrm{G}_{\alpha,\alpha-1,\frac{1}{2}}(-K,s)+KG_{\alpha,-1,\frac{1}{2}}(-K,\ s)\right]\mathrm{F}_{\alpha}\left(\frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}},\ t-s\right)\mathrm{d}s \tag{35}$$

In the case $\alpha=1$, lengthy but straightforward computations lead to the simpler expression, see (A9) from the Appendix:

$$\tau=\frac{\mathrm{Gr}}{\sqrt{\mathrm{Re}}(\sqrt{\mathrm{Pr}}+1)}\left\{\frac{2\sqrt{t}}{\sqrt{\pi}}+\frac{\sqrt{\mathrm{Pr}}}{\sqrt{\mathrm{Re}}}\left[1-\exp\left(\frac{\mathrm{Re}\,t}{\mathrm{Pr}}\right)\left(2-\mathrm{erfc}\frac{\sqrt{\mathrm{Re}\,t}}{\sqrt{\mathrm{Pr}}}\right)\right]\right\}+$$

$$+\frac{\mathrm{Gm}}{\sqrt{K\,\mathrm{Re}\,\mathrm{Sc}}}\left[\frac{1}{\sqrt{1-\mathrm{Sc}}}\exp\left(\frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}}t\right)\left(\mathrm{erfc}\sqrt{\frac{Kt}{1-\mathrm{Sc}}}-\mathrm{erf}\sqrt{\frac{K\,\mathrm{Sc}}{1-\mathrm{Sc}}t}\right)-\mathrm{erf}(\sqrt{Kt})\right] \tag{36}$$

Numerical results and discussions

In order to get some physical insight of present results, some numerical calculations have been carried out for different values of the fractional parameter α, the time t and physical parameters. However, in order to avoid repetition, only the most significant graphical representations regarding the effects of fractional parameter will be here included. The numerical values computed from analytical solutions of the problem have been visualized in figs. 1-6 both for $\alpha\in(0,1)$ and $\alpha=1$. As a confirmation of the validity of results that have been obtained, in all cases, the diagrams corresponding to fractional solutions tend to superpose over those of ordinary solutions when $\alpha\to 1$.

Figures 1 and 2 present the dimensionless temperature and concentration profiles at two times for different values of the fractional parameter α. As expected, the fluid temperature and concentration are increasing functions with respect to time. Their values, that are maxima near the plate, smoothly decrease to zero for increasing y. The influence of fractional parameter is significant and both the temperature and the concentration increase for increasing α. Furthermore, their values at any distance y from the plate are always higher for α_1 than those for α_2 if $\alpha_1 > \alpha_2$ and this result clearly confirms that for $0 < \alpha < 1$ the transport phenomenon exhibits sub-diffusion in comparison to the classical diffusion corresponding to $\alpha = 1$.

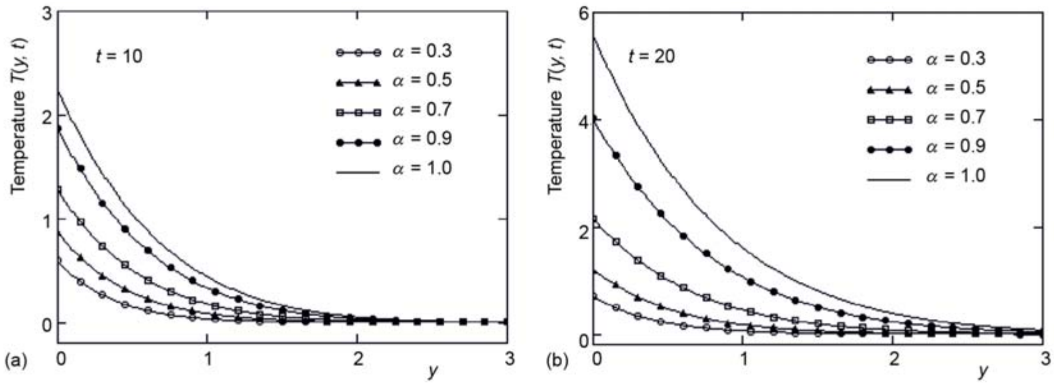

Figure 1. Temperature profiles *vs.* y for $\mathrm{Pr_{eff}} = 15$; (a) $t = 10$, (b) $t = 20$, and different values of α

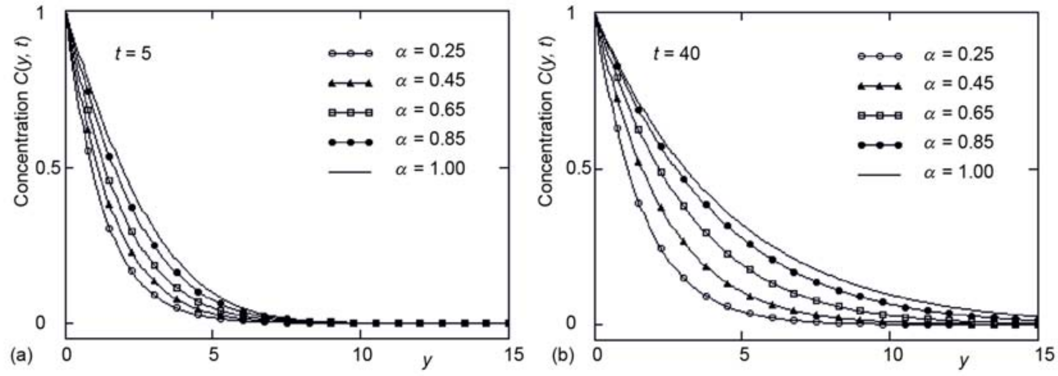

Figure 2. Concentration profiles *vs.* y for $\mathrm{Sc_{eff}} = 1$ and $K = 0.05$; (a) $t = 5$, (b) $t = 40$, and different values of α

The influence of effective Schmidt number $\mathrm{Sc_{eff}}$ and chemical reaction parameter K on the fluid concentration is presented in figs. 3 and 4. It is clearly seen from these figures that the concentration level of the fluid decreases whenever $\mathrm{Sc_{eff}}$ or K is increased. In the case of the effective Schmidt number, this is possible because an increase of $\mathrm{Sc_{eff}}$ means an increase of the Schmidt number that implies a fall in the mass diffusivity [12].

The dimensionless velocity profiles at two times for different values of α are depicted in figs. 5(a) and 5(b). It clearly results from these graphs that the fluid velocity against y is an increasing function with respect to t and α but a stronger increase appears with regard to the fractional parameter. Near the surface of the plate the fluid velocity increases, becomes maximum and then decreases to an asymptotic value for large values of y. In all cases the val

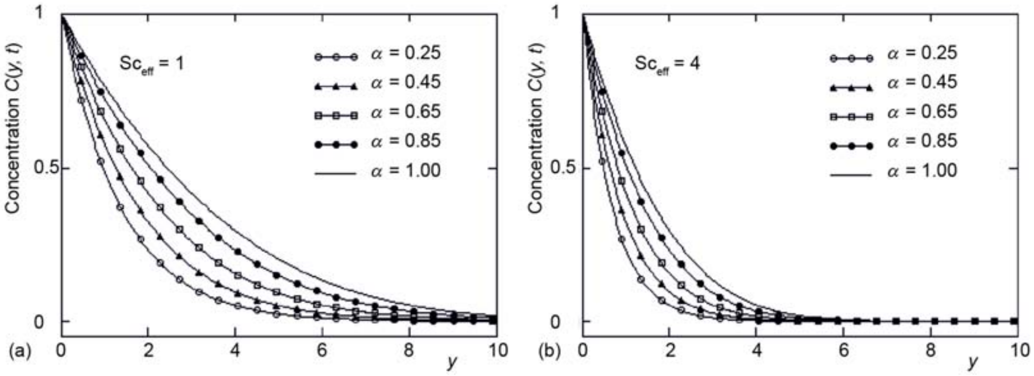

Figure 3. Concentration profiles *vs.* y for t = 10, K = 0.05, and Sc_{eff} = 1; (a) Sc_{eff} = 1, (b) Sc_{eff} = 4, and different values of α

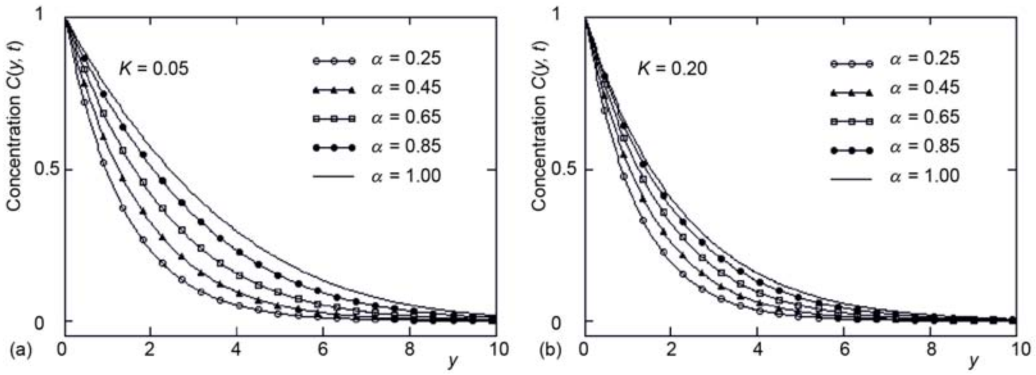

Figure 4. Concentration profiles *vs.* y for t = 10 and Sc_{eff} = 1; (a) K = 0.05, (b) K = 0.20, and different values of α

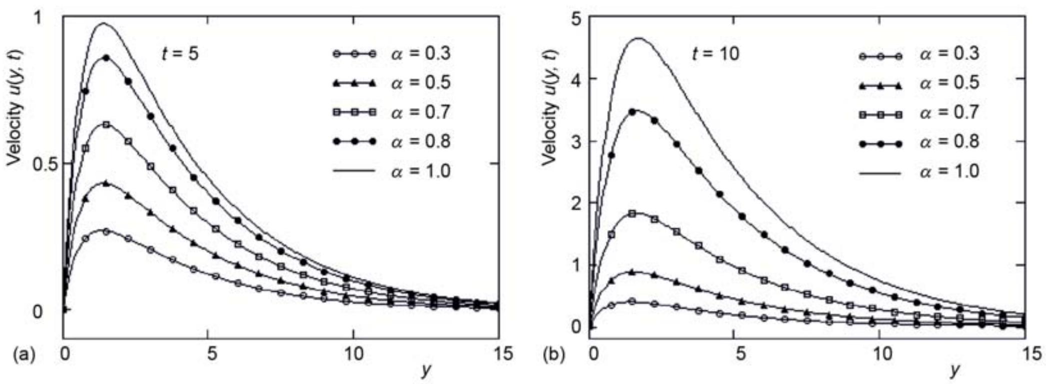

Figure 5. Velocity profiles *vs.* y for Re = 5, Pr = 25, Gr = 4, Gm = 0.3, Sc = 0.1, and K = 1.5; (a) t = 5, (b) t = 10, and different values of α

ues of velocity at any distance y from the plate are always higher or lower for distinct values of t or α. Figures 6(a) and 6(b) reveal the effects of the effective Schmidt number and of fractional parameter on the Sherwood number *vs.* t. The Sherwood number, as it results from these figures, is an increasing function with respect to Sc_{eff} and decreases in time from a maximum value near the plate to an asymptotic value for large values of the time. It also increases with respect to α up to a critical value of the time t (about 0.2) and decreases later.

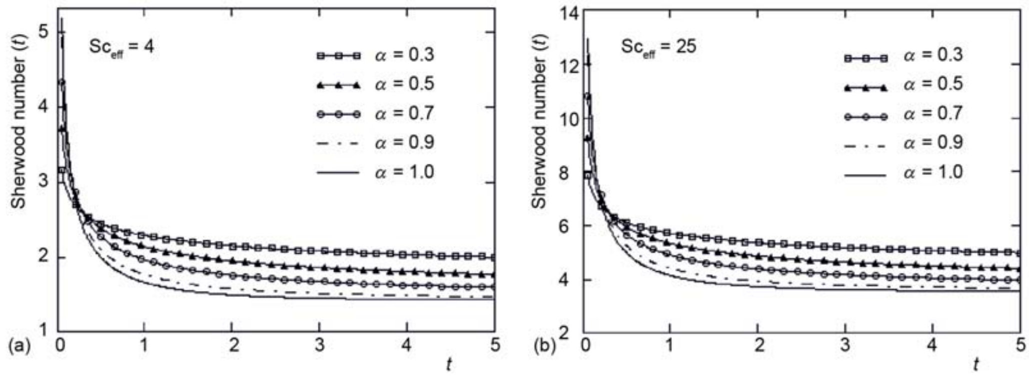

Figure 6. Sherwood number *vs.* t for K = 0.05; (a) Sc_{eff} = 4, (b) Sc_{eff} = 25, and different values of α

Finally, in order to have a clear idea about the accuracy of analytical solutions that have been here established in the fractional case, a comparison between the numerical and exact results was prepared for the dimensionless temperature and concentration. The corresponding results for the fluid temperature have been included in tab. 1. The temperature values resulting from eq. (18), where 35 terms of the sums have been taken into consideration, are compared with those obtained using the Stehfest's numerical algorithm for calculating the inverse Laplace transform [29]. This algorithm is based on the relation:

$$T(y,t) = L^{-1}\left[\overline{T}(y,q)\right] \approx \frac{\ln 2}{t}\sum_{j=1}^{2n} d_j \overline{T}\left(y, j\frac{\ln 2}{t}\right) \tag{37}$$

Table 1. Values of the dimensionless temperature $T_{an}(y)$ resulting from the analytic solution (18), and the numerical values $T_n(y)$ at t = 3, α = 0.65, and Pr_{eff} = 16

y	$T_{an}(y)$	$T_n(y)$	y	$T_{an}(y)$	$T_n(y)$	y	$T_{an}(y)$	$T_n(y)$
0.00	0.606959	0.606959	0.11	0.451100	0.451099	0.22	0.331601	0.331601
0.01	0.591073	0.591085	0.12	0.438852	0.438851	0.23	0.322278	0.322278
0.02	0.575550	0.575566	0.13	0.426899	0.426897	0.24	0.313189	0.313189
0.03	0.560383	0.560399	0.14	0.415233	0.415231	0.25	0.304329	0.304330
0.04	0.545566	0.545580	0.15	0.403849	0.403848	0.26	0.295694	0.295694
0.05	0.531092	0.531103	0.16	0.392747	0.392741	0.27	0.287278	0.287278
0.06	0.516955	0.516962	0.17	0.381906	0.381906	0.28	0.279076	0.279077
0.07	0.503148	0.503152	0.18	0.371336	0.371335	0.29	0.271085	0.271085
0.08	0.489666	0.489667	0.19	0.361025	0.361025	0.30	0.263299	0.263300
0.09	0.476500	0.476501	0.20	0.350969	0.350969			
0.10	0.463648	0.463647	0.21	0.341163	0.341163			

where n is a positive integer,

$$d_j = (-1)^{j+n} \sum_{k=\left[\frac{j+1}{2}\right]}^{\min(j,n)} \frac{k^n (2k)!}{(n-k)!k!(k-1)!(j-k)!(2k-j)!} \tag{38}$$

and $[r]$ denotes the integer part of the real number r. According to this table, the absolute error being of order 10^{-5}, there exists a very good agreement of analytical and numerical results.

Conclusions

The present work represents a theoretical study of the time-fractional free convection flow near a vertical plate with Newtonian heating, mass diffusion and chemical reaction. The radiative effects are not taken into consideration because, in the case of the Rosseland diffusion approximation [24], the heat transfer characteristics can be brought to light by means of a parameter only. More exactly, they can be included by re-scaling the effective Prandtl number to be $\mathrm{Pr}/[\mathrm{Re}(1+R)]$ where R is the radiation parameter. Consequently, a two parameter approach is superfluous.

The fractional model was firstly normalized and closed-form solutions for velocity, temperature, concentration, skin friction and the rates of heat and mass transfer from the plate to the fluid have been determined using the Laplace transform technique. It is worth pointing out that, in the absence of chemical reaction, the temperature and concentration depend on effective Prandtl number and the effective Schmidt number, respectively, which are transport parameters representing the thermal diffusivity and mass diffusivity. However, our interest here is on the special characteristics of the fractional model and the influence of fractional parameter α on the heat and mass transfer as well as on the fluid motion that are graphically underlined. The main findings are as follows.

- Exact solutions corresponding to the time-fractional free convection flow with Newtonian heating, mass diffusion and chemical reaction are established in terms of some known functions.
- The dimensionless temperature of the fluid and its concentration in the absence of chemical reaction depend of only one essential parameter $\mathrm{Pr}_{\mathrm{eff}}$ and $\mathrm{Sc}_{\mathrm{eff}}$, respectively.
- The fractional parameter α has a significant influence on the dimensionless temperature, concentration and velocity fields. They are increasing functions with respect to this parameter.
- The rate of mass transfer from the plate to fluid in terms of Sherwood number is an increasing function with regard to the effective Schmidt number and monotonically decreases in time.

Appendix

$$L^{-1}\left\{\frac{1}{q^{\mu}-a}\right\} = F_{\mu}(a,t) = \sum_{n=0}^{\infty} \frac{a^n t^{(n+1)\mu-1}}{\Gamma[(n+1)\mu]}; \quad \mu > 0 \tag{A1}$$

$$L^{-1}\left\{\frac{\mathrm{e}^{-aq^b}}{q^c}\right\} = t^{c-1}\Phi(c,-b,-at^{-b}); \quad \Phi(x,y,z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)\Gamma(x+ny)} \tag{A2}$$

$$L^{-1}\left\{\frac{q^{a-b}}{q^a+c}\right\}=t^{b-1}\mathrm{E}_{a,b}(-ct^a);\quad \mathrm{E}_{a,b}(z)=\sum_{k=0}^{\infty}\frac{zk}{\Gamma(ak+b)};\quad a>0,\, b>0 \tag{A3}$$

$$L^{-1}\left\{\frac{q^\alpha+a}{q}\right\}=L^{-1}\left\{\frac{1}{q^{1-\alpha}}+\frac{a}{q}\right\}=\begin{cases}\dfrac{t^{-\alpha}}{\Gamma(1-\alpha)}+a, & 0<\alpha<1\\ \delta(t)+a, & \alpha=1\end{cases} \tag{A4}$$

$$\text{If }\ \mathrm{F}(t)=L^{-1}[\mathrm{F}(q)]\ \text{and}\ \mathrm{g}(u,t)=L^{-1}\left[\mathrm{e}^{-uw(q)}\right]\ \text{then}\ L^{-1}\{\mathrm{F}[\mathrm{w}(q)]\}=\int_0^\infty \mathrm{f}(u)\mathrm{g}(u,t)\mathrm{d}u \tag{A5}$$

$$G_{a,b,c}(d,t)=L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}=\sum_{n=0}^{\infty}\frac{d^n\Gamma(n+c)t^{(n+c)a-b-1}}{\Gamma(c)\Gamma(n+1)\Gamma[(n+c)a-b]}$$

$$\text{if }\ \mathrm{Re}(ac-b)>0,\quad \mathrm{Re}(q)>0,\ \text{and}\ |d|<\left|q^a\right| \tag{A6}$$

$$L^{-1}\left\{\frac{\mathrm{e}^{-a\sqrt{q+b}}}{q-c}\right\}=\frac{\mathrm{e}^{ct}}{2}\left\{e^{-a\sqrt{b+c}}\mathrm{erfc}\left[\frac{a}{2\sqrt{t}}-\sqrt{(b+c)t}\right]+\right.$$

$$\left.+\mathrm{e}^{a\sqrt{b+c}}\mathrm{erfc}\left[\frac{a}{2\sqrt{t}}+\sqrt{(b+c)t}\right]\right\}=\Psi(a,b,c,t) \tag{A7}$$

$$L^{-1}\left\{\frac{e^{-a\sqrt{q}}}{q^2(\sqrt{q}+b)}\right\}=\frac{1}{b}\left(t+\frac{a^2}{2}+\frac{a}{b}+\frac{1}{b^2}\right)\mathrm{erfc}\frac{a}{2\sqrt{t}}-\frac{(2+ab)\sqrt{t}}{b^2\sqrt{\pi}}\exp\left(-\frac{a^2}{4t}\right)-$$

$$-\frac{1}{b^3}\exp(ab+b^2t)\,\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}+b\sqrt{t}\right)=\varphi(a,b,t) \tag{A8}$$

$$L^{-1}\left\{\frac{e^{-a\sqrt{q}}}{q(\sqrt{q}+b)}\right\}=\frac{1}{b}\mathrm{erfc}\frac{a}{2\sqrt{t}}-\frac{1}{b}e^{ab+b^2t}\,\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}+b\sqrt{t}\right) \tag{A9}$$

Nomenclature

C – concentration of the fluid, [kgm^{-3}]
C_p – specific heat at a constant pressure, [Jkg^{-1}K^{-1}]
C_w – concentration level at the plate, [kgm^{-3}]
C_∞ – concentration of the fluid far away from the plate, [kgm^{-3}]
D – mass diffusivity, [m^{2}s^{-1}]
g – acceleration due to gravity, [ms^{-2}]
Gm – mass Grashof number, [= $\gamma(C_w - C_\infty)$], [–]
Gr – thermal Grashof number, [= βT_∞] [–]
h – heat transfer coefficient, [Wm^{-2}K^{-1}]
K – chemical reaction parameter, [s^{-1}]
k – thermal conductivity of the fluid, [Wm^{-2}K^{-1}]
Nu – Nusselt number, [–]
Pr – Prandtl number (= μC_p/k), [–]
Sh – Sherwood number, [–]
T – temperature of the fluid, [K]
T_∞ – fluid temperature far away from the plate, [K]
u – velocity of the fluid along the *x*-axis, [ms^{-1}]

Greek symbols

β – the volumetric coefficient of thermal expansion, [K^{-1}]
γ – the volumetric coefficient of mass expansion, [m^{3}kg^{-1}]
μ – dynamic viscosity, [kgm^{-1}s^{-1}]
ν – kinematic viscosity, [m^{2}s^{-1}]
ρ – fluid density, [kgm^{-3}]
τ – skin friction, [Nm^{-2}]

$\mathrm{Pr_{eff}}$ – effective Prandtl number (= Pr/Re), [–]
Re – Reynolds number (= ν^2h^3/gk^3), [–]
Sc – Schmidt number (= ν/D), [–]
$\mathrm{Sc_{eff}}$ – effective Schmidt number (= Sc/Re), [–]

Subscripts

eff – effective
w – condition at the wall
∞ – free stream conditions

Acknowledgments

The authors would like to express their gratitude to reviewers for their careful assessment and fruitful suggestions regarding the first form of the manuscript.

References

[1] Ghoshdastidar, P. S., *Heat Transfer*, Oxford University Press, Oxford, UK, 2004
[2] Nield, D. A., Bejan, A., *Convection in Porous Media*, Springer, New York, USA, 2006
[3] Gebhart, B., Pera, L., The Nature of Vertical Natural Convection Flows Resulting from the Combined Buoyancy Effects of Thermal and Mass Diffusion, *International Journal of Heat and Mass Transfer*, *14* (1971), 2, pp. 2025-2050
[4] Chamkha, A. J., *et al.*, Radiation Effects on Free-Convection Flow past a Semi Infinite Vertical Plate with Mass Transfer, *Chemical Engineering Journal*, *84* (2001), 3, pp. 335-342
[5] Ganesan, P., Loganathan, P., Radiation and Mass Transfer Effects on Flow of an Incompressible Viscous Fluid past a Moving Cylinder, *International Journal of Heat and Mass Transfer*, *45* (2002), 21, pp. 4281-4288
[6] Cussler, E. L., *Diffusion Mass Transfer in Fluid Systems*, Cambridge University Press, London, 1988
[7] Mahapatra, M., *et al.*, Effects of Chemical Reaction on Free Convection Flow through a Porous Medium Bounded by a Vertical Surface, *Journal of Engineering Physics and Thermophysics*, *83* (2010), 1, pp. 130-140
[8] Sharma, P. R., *et al.*, Influence of Chemical Reaction and Radiation on Unsteady MHD Free Convection Flow and Mass Transfer through Viscous Incompressible Fluid past a Heated Vertical Plate Immersed in Porous Medium in the Presence of Heat Source, *Applied Mathematical Sciences*, *5* (2011), 46, pp. 2249-2260
[9] Muthucumaraswamy, R., Shankar, M. R., First Order Chemical Reaction and Thermal Radiation Effects on Unsteady Flow past an Accelerated Isothermal Infinite Vertical Plate, *Indian Journal of Science and Technology*, *4* (2011), 5, pp. 573-577
[10] Reddy, T. S., *et al.*, The Effects of Slip Condition, Radiation and Chemical Reaction on Unsteady MHD Periodic Flow of a Viscous Fluid through Saturated Porous Medium in a Planar Channel, *Journal on Mathematics*, *1* (2012), 1, pp. 18-28
[11] Reddy, T. S., *et al.*, Unsteady MHD Radiative and Chemically Reactive Free Convection Flow near a Moving Vertical Plate in Porous Medium, *Journal of Applied Fluid Mechanics*, *6* (2013), 3, pp. 443-451
[12] Ahmed, V., Dutta, M., Transient Mass Transfer Flow past an Impulsively Started Infinite Vertical Plate with Ramped Plate Velocity and Ramped Temperature, *International Journal of Physical Sciences*, *8* (2013), 7, pp. 254-263
[13] Reddy, T. S., *et al.*, Chemical Reaction and Radiation Effects on MHD Free Convection Flow through a Porous Medium Bounded by a Vertical Surface with Constant Heat and Mass Flux, *Journal of Computational and Applied Research*, *3* (2013), 1, pp. 53-62
[14] Srihari, K., Reddy, C. K., Effects of Soret and Magnetic Field on Unsteady Flow of a Radiating and Chemical Reacting Fluid: A Finite Difference Approach, *International Journal of Mechanical Engineering*, *3* (2014), 3, pp. 1-12
[15] Merkin, J. H., Natural Convection Boundary-Layer Flow on a Vertical Surface with Newtonian Heating, *International Journal of Heat and Fluid Flow*, *15* (1994), 5, pp. 392-398
[16] Narahari, M., *et al.*, Newtonian Heating and Mass Transfer on Free Convection Flow past an Accelerated Plate in the Presence of Thermal Radiation, *AIP Conference Proceedings*, *1482* (2012), 1, pp. 340-346
[17] Narahari, M., Dutta, B. K., Effects of Thermal Radiation and Mass Diffusion on Free Convection Flow near a Vertical Plate with Newtonian Heating, *Chemical Engineering Communications*, *199* (2012), 5, pp. 628-643

[18] Ramzan, M. *et al.*, MHD Three-Dimensional Flow of Couple Stress Fluid with Newtonian Heating, *European Physical Journal Plus*, *128* (2013), 5, No. 49
[19] Hussanan, A., *et al.*, Natural Convection Flow past an Oscillating Plate with Newtonian Heating, *Heat Transfer Research*, *45* (2014), 2, pp. 119-135
[20] Hussanan, A., *et al.*, Unsteady Boundary Layer MHD Free Convection Flow in a Porous Medium with Constant Mass Diffusion and Newtonian Heating, *European Physical Journal Plus*, *129* (2014), 3, No. 46
[21] Vieru, D., *et al.*, Magnetohydrodynamic Natural Convection Flow with Newtonian Heating and Mass Diffusion over an Infinite Plate that Applies Shear Stress to a Viscous Fluid, *Zeitschrift für Naturforschung A - Physical Sciences*, *69a* (2014), 12, pp. 714-724
[22] Heibig, A., Palade, L. I., On the Rest State Stability of an Objective Fractional Derivative Viscoelastic Fluid Model, *Journal of Mathematical Physics 49* (2008), 4, ID 043101-22
[23] Hatano, Y., *et al.*, Determination of Order in Fractional Diffusion Equation, *Journal of Math-for-Industry*, *5* (2013), A-7, pp. 51-57
[24] Magyari, E., Pantokratoras, A., Note on the Effect of Thermal Radiation in the Linearized Rosseland Approximation on the Heat Transfer Characteristics of Various Boundary Layer Flows, *International Communications in Heat and Mass Transfer*, *38* (2011), 5, 554-556
[25] Saha, U. K., *et al.*, On the Fractional Differintegration of Some Special Functions of Fractional Calculus and Related Functions, *International Journal of Mathematical and Computer Sciences*, *6* (2010), 2, pp. 65-69
[26] Stankovic, B., On the Function of E. M. Wright, *Publications de L'Institut Mathematique, Nouvelle serie*, *10* (1970), 24, pp. 113-124
[27] Hetnarski, R. B., An Algorithm for Generating Some Inverse Laplace Transforms of Exponential Form, *Z. Angew. Math. Phys.*, *26* (1975), 2, pp. 249-253
[28] Lorenzo, C. F., Hartley, T. T., Generalized Functions for the Fractional Calculus, *Critical Reviews™ in Biomedical Engineering*, *36* (2008), 1, pp. 39-55
[29] Stehfest, V., Algorithm 368: Numerical Inversion of Laplace Transform, *Communications of the ACM*, *13* (1970), 1, pp. 47-49

Paper submitted: November 15, 2014
Paper revised: February 2, 2015
Paper accepted: March 4, 2015