Open forum

INITIAL BOUNDARY VALUE PROBLEM FOR FRACTAL HEAT EQUATION IN THE SEMI-INFINITE REGION BY YANG-LAPLACE TRANSFORM

by

Yu-Zhu ZHANG [a,b], ***Ai-Min YANG*** [b,c,*], ***and Yue LONG*** [a]

[a] College of Metallurgy and Energy, Hebei United University, Tangshan, China
[b] College of Mechanical Engineering, Yanshan University, Qinhuangdao, China
[c] College of Science, Hebei United University, Tangshan, China

Short paper
DOI: 10.2298/TSCI130901152Z

Analytical solution of transient heat conduction through a semi-infinite fractal medium is developed. The solution focuses on application of a local fractional derivative operator to model the heat transfer process and a solution through the Yang-Laplace transform.

Key words: *initial boundary value problem, heat equation, fractal media, Yang-Laplace transform*

Introduction

Boundary value problems in heat conduction [1-4] are ever attractive due to their significant practical and academic aspects and commonly solved by numerical [5] or analytical [6] techniques such as variational iteration methods [7], Adomian's decomposition method [8] and the homotopy analysis method [9].

In case of fractional difference models of heat conduction [10-14] describing anomalous transport of thermal energy [15] in fractal media, the boundary value problems are described by fractional diffusion equations [16, 17] solved numerically or analytically [18-21]. Moreover, especially in the case of heat conduction, this leads to non-differentiable transport problems [22-27] solved in a variety ways, among them: local fractional variation iteration method [22, 23], local fractional Fourier series method [26], local fractional Laplace variational iteration method [27], *etc*.

This communication addresses a solution of transient heat conduction problem through a semi-infinite fractal medium [26] and developed by the Yang-Laplace transform. The Yang-Laplace transform [27-29] was conceived to solve some differential equations expressed through local fractional derivatives.

The mathematical method

For seek of clarity of the explanation, the properties for Yang-Laplace transform are briefly outlined. The Yang-Laplace transform is defined as [27-29]:

---

[*] Corresponding author; e-mail: aimin_heut@163.com

$$\tilde{L}_\alpha\{f(x)\} = f_s^{\tilde{L},\alpha}(s) = \frac{1}{\Gamma(1+\alpha)}\int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x)(\mathrm{d}x)^\alpha, \quad 0<\alpha\le 1 \tag{1}$$

and its inverse formula is defined as [27-29]:

$$f(x) = \tilde{L}_\alpha^{-1}\{f_s^{L,\alpha}(s)\} = \frac{1}{(2\pi)^\alpha}\int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{\tilde{L},\alpha}(s)(\mathrm{d}s)^\alpha \tag{2}$$

where $f(x)$ is the local fractional continuous, $s^\alpha = \beta^\alpha + i^\alpha\infty^\alpha$ and $\mathrm{Re}(s^\alpha) = \beta^\alpha$.

The following properties for Yang-Laplace transform are valid [28]:

$$\begin{gathered}\tilde{L}_\alpha\{f^{(\alpha)}(x)\} = s^\alpha \tilde{L}_\alpha\{f(x)\} - f(0), \quad \tilde{L}_\alpha\{x^{k\alpha}\} = \frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}}, \\ L_\alpha\{f(ax)\} = \frac{1}{a^\alpha} f_s^{L,\alpha}\left(\frac{s}{a}\right), \quad a>0\end{gathered} \tag{3a,b,c}$$

For more details of local fractional derivatives and integrals, see [26, 28].

Fractal heat equation in the semi-infinite region and its solution

The first law of thermodynamic states in fractal media reads as [26]:

$$\frac{1}{\Gamma(1+\alpha)}\int_{\tau_0}^{\tau}\left[\int_{V^{(\gamma)}}\iint\left(K^{2\alpha}\nabla^\alpha u + g - \rho_\alpha c_\alpha \frac{\partial^\alpha u}{\partial t^\alpha}\right)\mathrm{d}V^{(\gamma)}\right](\mathrm{d}t)^\alpha = 0 \tag{4a}$$

which leads to the equation [25, 26]:

$$K^{2\alpha}\nabla^\alpha u + g - \rho_\alpha c_\alpha \frac{\partial^\alpha u}{\partial t^\alpha} = 0 \tag{4b}$$

where the volume integral is the local fractional volume integral [26]. When the fractal dimension is equal to 1, eq. (4b) becomes the classical Fourier equation.

Making use of $g = 0$ and $\rho_\alpha c_\alpha = K^{2\alpha}$, the 1-D heat conduction through a semi-infinite fractal medium is modelled by [26, 27, 30]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0, \quad x>0, \quad t>0 \tag{5a}$$

$$u(x,0) = 0, \quad u(0,t) = u_0 \tag{5b,c}$$

with the Yang-Laplace transform, the model (5a, b, c) can be transformed into:

$$s^\alpha u(x,s) - u(x,0) - \frac{\partial^{2\alpha} u(x,s)}{\partial x^{2\alpha}} = 0, \quad u(0,s) = \frac{u_0}{s^\alpha} \tag{6a,b}$$

From eq. (6a), we get:

$$s^\alpha u(x,s) - \frac{\partial^{2\alpha} u(x,s)}{\partial x^{2\alpha}} = 0 \tag{7}$$

where the initial value condition is eq. (6b).

The general solution of eq. (7) can be expressed in the form:

$$u(x,s) = AE_\alpha(s^{\frac{\alpha}{2}} x^\alpha) + BE_\alpha(-s^{\frac{\alpha}{2}} x^\alpha) \tag{8}$$

In the expression (8) the pre-factors A and B are constants. However, taking into account that the temperature function is bounded, we get $A = 0$. Hence, from eq. (8) we have:

$$u(x,s)=\frac{u_0}{s^\alpha}E_\alpha(-s^{\frac{\alpha}{2}}x^\alpha) \tag{9}$$

Taking into account the transforms:

$$\tilde{L}_\alpha\{t^{2\alpha}f(t)\}=\frac{\partial^{2\alpha}T(s)}{\partial s^{2\alpha}},\quad \tilde{L}_\alpha\{-t^\alpha f(t)\}=\frac{\partial^\alpha T(s)}{\partial s^\alpha},\qquad \tilde{L}_\alpha\{f(t)\}=T(s)=E_\alpha(-s^{\frac{\alpha}{2}}) \tag{10 a,b,c}$$

we may obtain:

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha}+mf(t)=0,\quad m=\frac{4^\alpha t^\alpha\mu-1}{4^\alpha t^{2\alpha}},\quad \mu=\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}-\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{3\alpha}{2}\right)} \tag{10 d,e,f}$$

Following (10d) we have:

$$f(t)=\frac{1}{2^\alpha\Gamma\left(1-\frac{\alpha}{2}\alpha\right)t^{\frac{3\alpha}{2}}}E_\alpha\left[-\frac{\Gamma\left(1-\frac{5\alpha}{2}\right)}{4^\alpha\mu\Gamma\left(1-\frac{3\alpha}{2}\right)}\frac{1}{t^\alpha}\right] \tag{11}$$

Applying the Yang-Laplace transform we have:

$$\tilde{L}_\alpha^{-1}\left\{E_\alpha\left(-s^{\frac{\alpha}{2}}\right)\right\}\frac{1}{2^\alpha\Gamma\left(1-\frac{\alpha}{2}\right)t^{\frac{3\alpha}{2}}}E_\alpha\left[-\frac{\Gamma\left(1-\frac{5\alpha}{2}\right)}{4^\alpha\mu\Gamma\left(1-\frac{3\alpha}{2}\right)}\frac{1}{t^\alpha}\right] \tag{12a}$$

and

$$\tilde{L}_\alpha^{-1}\left\{\frac{E_\alpha\left(-s^{\frac{\alpha}{2}}x^\alpha\right)}{x^{2\alpha}s^\alpha}\right\}=\frac{1}{2^\alpha\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma(1+\alpha)}\int_0^t\frac{1}{\left(\frac{\tau}{x^2}\right)^{\frac{3\alpha}{2}}}E_\alpha\left[-\frac{\Gamma\left(1-\frac{5\alpha}{2}\right)}{4^\alpha\mu\Gamma\left(1-\frac{3\alpha}{2}\right)}\frac{1}{\left(\frac{\tau}{x^2}\right)^\alpha}\right](\mathrm{d}\tau)^\alpha \tag{12b}$$

From eq. (12b) we obtain the non-differentiable solution of eq. (5a) in the form:

$$u(x,t)=\frac{u_0x^{2\alpha}}{2^\alpha\Gamma\left(1-\frac{\alpha}{2}\alpha\right)\Gamma(1+\alpha)}\cdot$$
$$\cdot\int_0^t\frac{1}{\left(\frac{\tau}{x^2}\right)^{\frac{3\alpha}{2}}}E_\alpha\left\{\frac{\Gamma\left(1-\frac{5\alpha}{2}\right)}{\left[4^\alpha\left(\frac{\tau}{x^2}\right)^\alpha\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}-\frac{\Gamma\left(1-\frac{\alpha}{2}\right)}{\Gamma\left(1-\frac{3\alpha}{2}\right)}\right)\Gamma\left(1-\frac{3\alpha}{2}\right)\right]}\right\}(\mathrm{d}\tau)^\alpha \tag{13}$$

The solution (13) is a fractal function in accordance with the local fractional continuity concept [26].

Conclusions

The Yang-Laplace transform was successfully applied to solve an initial boundary value problem for fractal heat equation in the semi-infinite region, with local fractional derivatives and non-differentiable conditions. The result differs from those developed in [10-14] due to differences in the employed fractal operators. The solution allows, when the fractal dimension is $\alpha = \ln 2/\ln 3$ a solution on Cantor sets to be developed.

Acknowledgements

This work was supported by National scientific and technological support projects (No. 2012BAE09B00), the National Natural Science Foundation of China (No.11126213 and No. 61170317) and the National Natural Science Foundation of Hebei Province (No. A2012209043 and No. E2013209215).

Nomenclature

c_α – specific heat of fractal material, [Jkg^{-1}]
g – energy generation term
$K^{2\alpha}$ – thermal conductivity of the fractal material, [Wm$^{-\alpha}$K^{-1}]
$\tilde{L}_\alpha\{f(x)\}$ – Yang-Laplace transform of $f(x)$
$\tilde{L}_\alpha^{-1}[f_s^{L,\alpha}(s)]$ – inverse version of Yang-Laplace transform of $f_s^{L,\alpha}(s)$
t – time, [s]
$u(x.t)$ – temperature function, [K]
x – space co-ordinate, [m]

Greek symbols

a – fractal dimensional order, [–]
ρ_α – density, [kgm^{-3}]

References

[1] Oziisk, M. N., *Boundary Value Problems of Heat Conduction*, Dover Publishing, New York, USA, 1989
[2] Widder, D. V., *The Heat Equation*, Academic Press, New York, USA, 1976
[3] Chapko, R., *et al.*, An Inverse Boundary Value Problem for the Heat Equation: the Neumann Condition, *Inverse Problems, 15* (1999), 4, pp. 10-33
[4] Wei, T., Yamamoto, M., Reconstruction of a Moving Boundary from Cauchy Data in One-Dimensional Heat Equation, *Inverse Problems in Science and Engineering, 17* (2009), 4, pp. 551-567
[5] Douglas, J., Rachford, H. H., On the Numerical Solution of Heat Conduction Problems in Two and Three Space Variables, *Transactions of the American Mathematical Society, 82* (1956), 2, pp. 421-439
[6] Pruess, K., *et al.*, An Analytical Solution for Heat Transfer at a Boiling Front Moving through a Porous Medium, *International Journal of Heat and Mass Transfer, 30* (1987), 12, pp. 2595-2602
[7] Wu, G.-C., Laplace Transform Overcoming Princeple Drawbacks in Application of the Variational Iteration Method to Fractional Heat Equations, *Thermal Science, 16* (2012), 4, pp. 1257-1261
[8] Pamuk, S., An Application for Linear and Nonlinear Heat Equations by Adomian's Decomposition Method, *Applied Mathematics and Computation, 163* (2005), 1, pp. 89-96
[9] Abbasbandy, S., The Application of Homotopy Analysis Method to Nonlinear Equations Arising in Heat Transfer, *Physics Letters A, 360* (2006). 1, pp. 109-113
[10] Agarwal, R. P., *et al.*, Boundary Value Problems for Fractional Differential Equations, *Georgian Mathematical Journal, 16* (2009), 3, pp. 401-411
[11] Benchohra, M., *et al.*, Boundary Value Problems for Differential Equations with Fractional Order and Nonlocal Conditions, *Nonlinear Analysis: Theory, Methods & Applications, 71* (2009), 7, pp. 2391-2396
[12] Wang, Q. L., *et al.*, Fractional Model for Heat Conduction in Polar Bear Hairs, *Thermal Science, 16* (2012), 2, pp. 339-342
[13] Fan, J., J. He, H., Biomimic Design of Multi-Scale Fabric with Efficient Heat Transfer Property, *Thermal Science, 16* (2012), 5, pp. 1349-1352

[14] Li, Z. B., *et al*., Exact Solutions of Time-Fractional Heat Conduction Equation by the Fractional Complex Transform, *Thermal Science, 16* (2012), 2, pp. 335-338
[15] Metzler, R., Klafter, J., Boundary Value Problems for Fractional Diffusion Equations, *Physica A: Statistical Mechanics and its Applications, 278* (2000), 1, pp. 107-125
[16] Luchko, Y., Initial-Boundary-Value Problems for the Generalized Multi-Term Time-Fractional Diffusion Equation, *Journal of Mathematical Analysis and Applications, 374* (2011), 2, pp. 538-548
[17] Povstenko, Y. Z., Fractional Radial Heat Conduction in an Infinite Medium with a Cylindrical Cavity and Associated Thermal Stresses, *Mechanics Research Communications, 37* (2010), 4, pp. 436-440
[18] Scherer, R., *et al*., Numerical Treatment of Fractional Heat Equations, *Applied Numerical Mathematics, 58* (2008), 8, pp. 1212-1223
[19] Hristov, J., Approximate Solutions to Fractional Sub-Diffusion Equations: The Heat-Balance Integral Method, *The European Physical Journal Special Topics, 193* (2011), 4, pp. 229-243
[20] Hristov, J., Heat-Balance Integral to Fractional (Half-Time) Heat Diffusion Sub-Model, *Thermal Science, 14* (2010), 2, pp. 291-316
[21] Hristov, J., Transient Flow of a Generalized Second Grade Fluid Due to a Constant Surface Shear Stress: An Approximate Integral-Balance Solution, *Int. Rev. Chem. Eng, 3* (2011), 6, pp. 802-809
[22] Yang, X.-J., Baleanu, D., Fractal Heat Conduction Problem Solved by Local Fractional Variation Iteration Method, *Thermal Science, 17* (2013), 2, pp. 625-628
[23] He, J.-H., Liu, F.-J., Local Fractional Variational Iteration Method for Fractal Heat Transfer in Silk Cocoon Hierarchy, *Nonlinear Science Letters A, 4* (2013), 1, pp. 15-20
[24] Zhang, Y.-Z, *et al*., 1-D Heat Conduction in a Fractal Medium: A Solution by the Local Fractional Fourier series Method, *Thermal Science, 17* (2013), 3, pp. 953-956
[25] Yang, X.-J., Heat Transfer in Discontinuous Media, *Advances in Mechanical Engineering and its Applications, 1* (2012), 3, pp. 47-53
[26] Yang, X.-J., Advanced Local Fractional Calculus and Its Applications, World Science Publisher, New York, USA, 2012
[27] Liu, C. F., *et al.*, Reconstructive Schemes for Variational Iteration Method within Yang-Laplace Transform with Application to Fractal Heat Conduction Problem, *Thermal Science, 17* (2013), 3, pp. 715-721
[28] Yang, X.-J., *Local Fractional Functional Analysis and Its Applications*, Asian Academic Publisher Limited, Hong Kong, 2011
[29] He, J.-H., Asymptotic Methods for Solitary Solutions and Compactons, Abstract and Applied Analysis, 2012 (2012), Article ID 916793
[30] Yang, A.-M., *et al*., Local Fractional Series Expansion Method for Solving Wave and Diffusion Equations on Cantor Sets, *Abstract and Applied Analysis, 2013* (2013), Article ID 351057, 8

Paper submitted: September 1, 2013
Paper revised: November 8, 2013
Paper accepted: November 8, 2013