

## NEW METHODS TO PROVIDE EXACT SOLUTIONS FOR SOME UNIDIRECTIONAL MOTIONS OF RATE TYPE FLUIDS

by

**Corina FETECAU<sup>a\*</sup>, Qammar RUBBAB<sup>b</sup>, Shahraz AKHTER<sup>c</sup>,  
and Constantin FETECAU<sup>d,e</sup>**

<sup>a</sup>Department of Theoretical Mechanics, Technical University of Iasi, Iasi, Romania

<sup>b</sup>Department of Mathematics, Air University, Islamabad, Pakistan

<sup>c</sup>Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan

<sup>d</sup>Department of Mathematics, Technical University of Iasi, Iasi, Romania

<sup>e</sup>Academy of Romanian Scientists, Bucuresti, Romania

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*Based on three immediate consequences of the governing equations corresponding to some unidirectional motions of rate type fluids, new motion problems are tackled for exact solutions. For generality purposes, exact solutions are developed for shear stress boundary value problems of generalized Burgers fluids. Such solutions, for which the shear stress instead of its differential expressions is given on the boundary, are lack in the literature for such fluids. Consequently, the first exact solutions for motions of rate type fluids induced by an infinite plate or a circular cylinder that applies a constant shear  $f$  or an oscillating shear  $f\sin(\omega t)$  to the fluid are here presented. In addition, all steady-state solutions can easily be reduced to known solutions for second grade and Newtonian fluids.*

*Key words: rate type fluids, unidirectional motions, shear stress boundary value problems, exact solutions*

### Introduction

Flows over an infinite plate or through a circular cylinder are extensively studied in the literature, they being some of the most important motion problems near moving bodies with many practical applications. There are a lot of exact solutions corresponding to such motions of Newtonian or non-Newtonian fluids. In the last time, some of them have been also extended to non-Newtonian fluids with fractional derivatives [1-3]. However, the most part of them corresponds to motion problems for which the velocity is given on the boundary although in some practical problems what is specified is the shear stress on the boundary [4, 5]. Renardy [4], for instance, considers the flow of a Maxwell fluid across a strip bounded by parallel planes and shows that boundary conditions on stresses at the inflow boundary have to be imposed in order to formulate a well-posed boundary value problem. He also considers steady flows of viscoelastic fluids [6] and shows that the Jeffrey model is well-posed in a bounded channel if all components of the extra-stress are given on the entry boundary. Furthermore, the *no slip* boundary condition may not be necessarily applicable for flows of polymeric fluids that can slip or

\* Corresponding author; e-mail: c\_fetecau@yahoo.com

slide on the boundary. Consequently, the boundary conditions on stresses are particularly meaningful and Renardy [5] demonstrated how well-posed boundary value problems can be formulated in this fashion. Contrary to what is usually assumed that one can prescribe the velocity of a moving plate, as far as the physical situation is concerned one can only prescribe the force with which the plate is moved. How the plate moves is a consequence of this applied force. One needs to use transducers and feedback control mechanisms and constantly adjust the force in order to maintain the velocity. To reiterate, in Newtonian mechanics force is the cause and kinematics is the effect (see [7] for a detailed discussion of the same). Prescribing the shear stress at the plate is tantamount to prescribing the (shear) force applied to the plate.

To the best of our knowledge, the first exact solutions for motions of rate type fluids in which the shear stress is given on the boundary are those of Waters and King [8]. In the last time many similar solutions for such motions have been established ([9-14] and references therein). However, it is worth pointing out that all these solutions correspond to motion problems for which differential expressions of shear stresses are given on the boundary. This is due to their governing equations that, unlike those corresponding to Newtonian and second grade fluids, contain differential expressions acting on the stresses. Consequently, in the literature, does not exist exact solutions corresponding to motions of rate type fluids due to a constant shear  $f$  or an oscillating  $f\sin(\omega t)$  or  $f\cos(\omega t)$  shear stress on the boundary. Such solutions are known only for Newtonian and second grade fluids [15-19].

The aim of this note is to remove this drawback. To do that, three simple remarks regarding the governing equations corresponding to some unidirectional motions of rate type fluids are brought to light and carefully used. More exactly, these remarks refer to some unidirectional motions of rate type fluids induced by a plate or an infinite circular cylinder that are moving or apply a given shear stress to the fluid. In order to illustrate the power of these remarks, as well as for the generality of solutions, we provide exact solutions for motions of incompressible generalized Burgers fluids (IGBF) due to constant, uniformly accelerated or oscillating shear stresses on the boundary. These solutions, occasionally obtained using known solutions from the literature, can easily be reduced to the similar solutions for Burgers fluids. As a check of general results, the steady-state solutions corresponding to oscillating motions are used to recover known solutions obtained in the literature by a different technique for second grade and Newtonian fluids.

### Constitutive and governing equations

Many models are used to describe the rheological behavior of non-Newtonian fluids. Among them, the rate type fluids are those which take into account elastic and memory effects. The simplest subclasses of rate type fluids are those of Maxwell and Oldroyd-B fluids. However, these fluids do not exhibit rheological properties of many real fluids such as the asphalt in geomechanics and cheese in food products. Relatively recent, Rajagopal and Srinivasa [20] generalized these models to the Burgers model that has been successfully used to describe the motion of the earth's mantle and the response of asphalt and asphalt concrete [21]. It is also used to model different geological structures, such as Olivine rocks [22] and the propagation of seismic waves in the interior of the Earth [23]. An immediate and natural generalization of Burgers model is that of the generalized Burgers fluids (GBF) [24-27].

The stress tensor  $\mathbf{T}$  for IGBF is related to the fluid motion in the manner:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda_1 \frac{\delta \mathbf{S}}{\delta t} + \lambda_2 \frac{\delta^2 \mathbf{S}}{\delta t^2} = \mu \left( \mathbf{A} + \lambda_3 \frac{\delta \mathbf{A}}{\delta t} + \lambda_4 \frac{\delta^2 \mathbf{A}}{\delta t^2} \right) \quad (1)$$

where  $p\mathbf{I}$  denotes the indeterminate spherical stress,  $\mathbf{S}$  is the extra-stress tensor,  $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$  – the first Rivlin-Ericksen tensor,  $\mathbf{L}$  – the velocity gradient,  $\mu$  – the dynamic viscosity,  $\lambda_1$  and  $\lambda_3 (< \lambda_1)$  are relaxation and retardation times, respectively,  $\lambda_2$  and  $\lambda_4$  are new material constants having the dimensions of the square of time, and  $\delta/\delta t$  denotes the upper convected derivative defined by:

$$\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T \quad (2)$$

Of course, this model contains as special cases: Burgers fluids for  $\lambda_4 = 0$ , Oldroyd-B fluids for  $\lambda_2 = \lambda_4 = 0$ , Maxwell fluids for  $\lambda_2 = \lambda_3 = \lambda_4 = 0$ , and Newtonian fluids for  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ . For flows to be here considered, the governing equations corresponding to GBF also resemble those for second grade fluids. Consequently, our remarks regarding the differential equations governing some unidirectional motions of IGBF are also valid for all mentioned fluids whose behavior can be approximated by means of general solutions taking small values for some material constants. Furthermore, the steady-state solutions can be particularized to give the corresponding solutions for Burgers, Oldroyd-B, Maxwell, second grade, and Newtonian fluids. The starting solutions can be reduced only to the similar solutions for Burgers fluids.

#### *Unidirectional motions over an infinite plate*

Let us assume that an IGBF is at rest over an infinite flat plate which is situated in the plane  $y = 0$  of a fixed Cartesian co-ordinate system  $x$ ,  $y$ , and  $z$ . The plate is moving in its plane with a given velocity or applies a shear stress to the fluid after time  $t = 0^+$ . Due to the shear the fluid is gradually moved and its velocity is of the form:

$$\vec{v} = \vec{v}(y, t) = u(y, t) \vec{i} \quad (3)$$

where  $\vec{i}$  is the unit vector along the  $x$ -direction. For such flows the constraint of incompressibility is automatically satisfied. We also assume that the extra-stress tensor  $\mathbf{S}$ , as well as the velocity  $\vec{v}$ , depends on  $y$  and  $t$  only.

Introducing the velocity field (3) in the constitutive eq. (1)<sub>2</sub> and bearing in mind that the fluid is at rest at the moment  $t = 0$  we find that the non-trivial shear stress  $\tau(y, t) = S_{xy}(y, t)$  satisfies the partial differential equation [24, eq. (10)]:

$$\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) \tau(y, t) = \mu \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \frac{\partial u(y, t)}{\partial y} \quad (4)$$

In the absence of body forces and a pressure gradient in the flow direction, the balance of linear momentum reduces to the meaningful equation:

$$\frac{\partial \tau(y, t)}{\partial y} = \rho \frac{\partial u(y, t)}{\partial t} \quad (5)$$

By eliminating  $\tau(y, t)$  between eqs. (4) and (5), as usually in the literature, one obtains the governing equation for velocity:

$$\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) \frac{\partial u(y, t)}{\partial t} = \nu \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \frac{\partial^2 u(y, t)}{\partial y^2} \quad (6)$$

Here  $\nu = \mu/\rho$  is the kinematic viscosity of the fluid and  $\rho$  is its constant density.

In order to solve well-posed initial boundary value problems, for such motions, the authors use eqs. (4) and (6) to determine the velocity and shear stress distributions. This is the rea-

son that all exact solutions that have been obtained for such motions of rate type fluids, when the shear stress is given on the boundary, correspond to differential expressions of the shear stress on the boundary. This is not true for Newtonian or second grade fluids because eq. (4) reduces to:

$$\tau(y, t) = \mu \frac{\partial u(y, t)}{\partial y}, \text{ respectively } \tau(y, t) = \mu \left( 1 + \lambda_3 \frac{\partial}{\partial t} \right) \frac{\partial u(y, t)}{\partial y} \quad (7)$$

Our main interest here is to provide exact solutions for motions of rate type fluids induced by an infinite plate that applies a constant shear  $f$  or an oscillating shear  $f \sin(\omega t)$  to the fluid. Such solutions are similar to those corresponding to Stokes' problems when the plate is moving in its plane with a constant velocity  $V$  or oscillates according to  $V \sin(\omega t)$  and are identically important. In order to do that, let us eliminate the velocity  $u(y, t)$  between eqs. (4) and (5). The result:

$$\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) \frac{\partial \tau(y, t)}{\partial t} = \nu \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \frac{\partial^2 \tau(y, t)}{\partial y^2} \quad (8)$$

shows that the shear stress  $\tau(y, t)$  has to satisfy a linear partial differential equation of the same form as velocity. This simple but unexpected result, as we shall later see, can have crucial applications in the field. It allows us to solve new motion problems and to provide exact solutions using known results from the literature.

#### *Unidirectional motions produced by an infinite circular cylinder*

Such motions can be generated, for instance, by an infinite circular cylinder that is moving along or around its axis or applies longitudinal or rotational shear stresses to the fluid. For these motions we are looking for a velocity field  $\vec{v}(r, t)$  of the form:

$$\vec{v} = \vec{v}(r, t) = w(r, t) \vec{e}_\theta \quad \text{or} \quad \vec{v} = \vec{v}(r, t) = v(r, t) \vec{e}_z \quad (9)$$

where  $\vec{e}_\theta$  and  $\vec{e}_z$  are unit vectors in the  $\theta$ - and  $z$ -directions of a cylindrical co-ordinate system  $r$ ,  $\theta$ , and  $z$ . The constraint of incompressibility is identically satisfied for both motions. We again assume that the extra-stress tensor  $\mathbf{S}$ , as well as the velocity, depends on  $r$  and  $t$  only.

Assuming that the fluid is at rest until the moment  $t = 0$  and separately introducing eq. (9) in eq. (1)<sub>2</sub>, we get the next meaningful partial differential equations (see [27], in eqs. (8a) and (8b)], for instance):

$$\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) \tau_w(r, t) = \mu \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \left( \frac{\partial}{\partial r} - \frac{1}{r} \right) w(r, t) \quad (10)$$

$$\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) \tau_v(r, t) = \mu \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \frac{\partial v(r, t)}{\partial r} \quad (11)$$

where  $\tau_w(r, t) = \mathbf{S}_{r\theta}(r, t)$  and  $\tau_v(r, t) = \mathbf{S}_{rz}(r, t)$  are the non-trivial shear stresses corresponding to the two motions.

Neglecting body forces and in absence of a pressure gradient in the axial direction, the balance of linear momentum reduces to the meaningful partial differential equations ( $\partial p / \partial \theta$  is zero due to the rotational symmetry):

$$\rho \frac{\partial w(r, t)}{\partial t} = \left( \frac{\partial}{\partial r} + \frac{2}{r} \right) \tau_w(r, t), \text{ respectively } \rho \frac{\partial v(r, t)}{\partial t} = \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) \tau_v(r, t) \quad (12)$$

Following the same line from the literature, we eliminate  $\tau_w(r, t)$  and  $\tau_v(r, t)$  between eqs. (10) and (12)<sub>1</sub>, respectively (11) and (12)<sub>2</sub>. The resulting partial differential equations, namely:

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial w(r, t)}{\partial t} = \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r, t) \quad (13)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial v(r, t)}{\partial t} = \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}\right) v(r, t) \quad (14)$$

are the governing equations for the components  $w(r, t)$  and  $v(r, t)$  of velocity.

It is worth pointing out that for all exact solutions that have been established for such unidirectional motions of rate type fluids, the governing eqs. (10) and (13) or (11) and (14) have been used. Due to this fact the solutions for stress boundary value problems correspond to differential expressions of the shear stress on the boundary, excepting those for second grade and Newtonian fluids [15-19]. In the case of Newtonian fluids, for instance, eqs. (10) and (11) take the simple forms:

$$\tau_w(r, t) = \mu \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w(r, t), \quad \tau_v(r, t) = \mu \frac{\partial v(r, t)}{\partial r} \quad (15)$$

and a motion due to a constant shear  $f$ , a uniformly accelerated shear  $ft$  or an oscillating shear  $f \sin(\omega t)$  on the boundary can easily be studied for such fluids.

In order to be able to develop exact solutions for motions of rate type fluids produced by a circular cylinder that applies constant or oscillating shear stresses to the fluid, we eliminate the velocities  $w(r, t)$  and  $\bar{v}(r, t)$  between eqs. (10) and (13), respectively (11) and (14). The results:

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial \tau_w(r, t)}{\partial t} = \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{4}{r^2}\right) \tau_w(r, t) \quad (16)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2}\right) \frac{\partial \tau_v(r, t)}{\partial t} = \nu \left(1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) \tau_v(r, t) \quad (17)$$

show that the shear stress  $\tau_v(r, t)$  satisfies a partial differential equation of the same form as the component  $w(r, t)$  of velocity while the shear stress  $\tau_w(r, t)$  satisfies a different partial differential equation whose resolution is simple enough. They help us to develop new exact solutions using known results from the literature or to solve motion problems with rotational or longitudinal shear stresses prescribed on a rigid circular cylinder.

### Applications

In order to bring to light the theoretical and practical value of the governing eqs. (8), (16), and (17) for the non-trivial shear stresses  $\tau(y, t)$ ,  $\tau_w(r, t)$ , and  $\tau_v(r, t)$ , we shall consider some motion problems whose solution is lack in the literature for rate type fluids. For these problems, we shall prescribe the values of the shear stresses on the boundary instead of certain differential expressions of them.

#### *Flow induced by an infinite plate that applies a constant shear to the fluid*

Let us consider an IGBF at rest over an infinite plate situated in the  $(x, z)$  plane. At time  $t = 0^+$  the plate is pulled with a constant shear  $f$  along the  $x$ -axis. Owing to the shear the fluid is

gradually moved. Its velocity  $u(y, t)$  and the non-trivial shear stress  $\tau(y, t)$  satisfy the partial differential eqs. (4) and (6), respectively (5) and (8). The appropriate initial and boundary conditions are:

$$u(y, 0) = 0, \quad \tau(y, 0) = \frac{\partial \tau(y, 0)}{\partial t} = \frac{\partial^2 \tau(y, 0)}{\partial t^2} = 0 \quad \text{for } y > 0 \quad (18)$$

$$\tau(0, t) = fH(t) \quad \text{for } t \geq 0 \quad (19)$$

where  $H(\cdot)$  is the Heaviside step function. Furthermore, the natural condition:

$$\tau(y, t) \rightarrow 0 \quad \text{as } y \rightarrow \infty \quad \text{and } t \geq 0 \quad (20)$$

has also to be satisfied. It shows us that there is no shear in the free stream.

The solution of linear partial differential eq. (8) with the corresponding initial and boundary conditions (18)-(20), as it results from [14, eq. (3.7)], has the simple form:

$$\begin{aligned} \tau(y, t) = fH(t) - \frac{2f}{\pi\lambda_2} H(t) \int_0^\infty & \left[ \frac{\lambda_1 q_1^2 + \lambda_1 q_1 + 1}{(q_1 - q_2)(q_1 - q_3)} e^{q_1 t} + \frac{\lambda_2 q_2^2 + \lambda_1 q_2 + 1}{(q_2 - q_1)(q_2 - q_3)} e^{q_2 t} + \right. \\ & \left. + \frac{\lambda_2 q_3^2 + \lambda_1 q_3 + 1}{(q_3 - q_1)(q_3 - q_2)} e^{q_3 t} \right] \frac{\sin(y\xi)}{\xi} d\xi \end{aligned} \quad (21)$$

where  $q_i = s_i - (\lambda_1 + \beta\xi^2)/(3\lambda_2)$  with  $i = 1, 2, 3$  are the roots of the algebraic equation:  $\lambda_2 q^3 + (\lambda_1 + \beta\xi^2)q^2 + (1 + \alpha\xi^2)q + \nu\xi^2 = 0$ ;  $\alpha = \nu\lambda_3, \beta = \nu\lambda_4$ .

Into above relations (see Cardano's formulae [28]):

$$\begin{aligned} s_1 = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \quad s_2 = z^3 \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + z^2 \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}, \\ s_3 = z^2 \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + z^3 \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \end{aligned} \quad (22)$$

are the roots of the algebraic equation  $s^3 + as + b = 0$ , where  $a = [(1 + \alpha\xi^2)/\lambda_2] - [(\lambda_1 + \beta\xi^2)/(3\lambda_2^2)]$ ,  $b = (\nu\xi^2/\lambda_2) + [(2(\lambda_1 + \beta\xi^2)^3)/(27\lambda_2^3)] - [(\lambda_1 + \beta\xi^2)(1 + \alpha\xi^2)]/(3\lambda_2^2)$ , and  $z = [(-1 + i(3)^{1/2})/2]$ .

Introducing eq. (21) into eq. (5), integrating with respect to time from 0 to  $t$  and using the initial condition (18)<sub>1</sub>, we find the corresponding velocity  $u(y, t)$  under the form:

$$\begin{aligned} u(y, t) = \frac{2f}{\rho\pi\lambda_2} \int_0^\infty & \left[ \frac{\lambda_2 q_1^2 + \lambda_1 q_1 + 1}{q_1(q_1 - q_2)(q_1 - q_3)} (1 - e^{q_1 t}) + \frac{\lambda_2 q_2^2 + \lambda_1 q_2 + 1}{q_2(q_2 - q_1)(q_2 - q_3)} (1 - e^{q_2 t}) + \right. \\ & \left. + \frac{\lambda_2 q_3^2 + \lambda_1 q_3 + 1}{q_3(q_3 - q_1)(q_3 - q_2)} (1 - e^{q_3 t}) \right] \cos(y\xi) d\xi \end{aligned} \quad (23)$$

By now letting  $\lambda_4 \rightarrow 0$  into eqs. (21) and (23), the similar solutions for Burgers' fluids are obtained. The corresponding solutions for Oldroyd-B and Maxwell fluids can be immediately established using [14, eqs. (4.1) and (4.4)].

The shear stress corresponding to the motion of an Oldroyd-B fluid due to an infinite plate that applies a constant shear  $f$  to the fluid, for instance, is:

$$\tau_{\text{OB}}(y, t) = fH(t) - \frac{2f}{\pi\lambda} H(t) \int_0^\infty \frac{(\lambda q_4 + 1)e^{q_4 t} - (\lambda q_5 + 1)e^{q_5 t}}{q_4 - q_5} \frac{\sin(y\xi)}{\xi} d\xi \quad (24)$$

where  $q_4, q_5 = \{-(1 + \alpha\xi^2) \mp [(1 + \alpha\xi^2)^2 - 4v\lambda\xi^2]^{1/2}\}/(2\lambda)$  and  $\lambda = \lambda_1$ . By making into eq. (24), we recover the result:

$$\tau_{SG}(y, t) = fH(t) \left[ 1 - \frac{2}{\pi} \int_0^\infty \frac{1}{1 + \alpha\xi^2} \exp\left(-\frac{v\xi^2}{1 + \alpha\xi^2} t\right) \frac{\sin(y\xi)}{\xi} d\xi \right] \quad (25)$$

obtained from [9, eq. (4.1)<sub>2</sub>] by a different technique. The expression (25) of  $\tau_{SG}(y, t)$ , in a perfect accordance with [29, eq. (3)], represents the shear stress distribution corresponding to the motion of a second grade fluid induced by an infinite plate that applies a constant shear  $f$  to the fluid. Introducing eq. (25) into eq. (5), integrating with respect to  $t$  and using the corresponding initial condition we recover [9, eq. (4.1)<sub>1</sub>]:

$$v_{SG}(y, t) = \frac{fy}{\mu} - \frac{2f}{\mu\pi} \int_0^\infty \left[ 1 - \exp\left(-\frac{v\xi^2}{1 + \alpha\xi^2} t\right) \right] \cos(y\xi) \frac{1}{\xi^2} d\xi \quad (26)$$

#### Flow over an infinite plate that applies oscillating shears to fluid

Let us again suppose that an IGBF is at rest over an infinite plate. After time  $t = 0^+$  the plate applies an oscillating shear stress  $f\sin(\omega t)$  or  $f\cos(\omega t)$  to the fluid. Due to the shear the fluid begins to move. Its velocity and shear stress satisfy the same governing equations as before. The initial conditions are also the same while the boundary condition (19) becomes:

$$\tau(0, t) = f \sin(\omega t) \quad \text{or} \quad \tau(0, t) = fH(t) \cos(\omega t) \quad \text{for} \quad t \geq 0 \quad (27)$$

The starting solutions corresponding to oscillating motions of fluids are generally presented as a sum of steady-state and transient solutions. They describe the motion of the fluid some time after its initiation. After that time, when the transients disappear, the fluid moves according to the steady-state solutions which are periodic in time and independent of the initial conditions. However, they satisfy the boundary conditions and governing equations.

In view of eq. (8), the steady-state components of the shear stress  $\tau(y, t)$  corresponding to the sine and cosine oscillations of the shear on the boundary can be written in the simple forms [24, eqs. (22)]:

$$\tau_{ss}(y, t) = fe^{-my} \sin(\omega t - ny) \quad \text{and} \quad \tau_{cs}(y, t) = fe^{-my} \cos(\omega t - ny) \quad (28)$$

where [24, eqs. (23) and (24)]

$$m^2, n^2 = \frac{\omega}{2v} \frac{\sqrt{[(1 - \lambda_4\omega^2)^2 + (\lambda_3\omega)^2][(1 - \lambda_2\omega^2)^2 + (\lambda_1\omega)^2]} \pm \omega[\lambda_3 - \lambda_1 + (\lambda_1\lambda_4 - \lambda_2\lambda_3)\omega^2]}{(1 - \lambda_4\omega^2)^2 + (\lambda_3\omega)^2}$$

Introducing eqs. (28) into eq. (5), integrating with respect to  $t$  and bearing in mind the time periodicity of the steady-state solutions, we find that:

$$u_{ss}(y, t) = \left( \frac{f}{\rho\omega} \right) \sqrt{m^2 + n^2} e^{-my} \cos(\omega t - ny + \varphi) \quad (29)$$

$$u_{cs}(y, t) = \left( \frac{f}{\rho\omega} \right) \sqrt{m^2 + n^2} e^{-my} \cos\left[\omega t - ny + \varphi + \left(\frac{\pi}{2}\right)\right]$$

where  $\text{tg}\varphi = m/n$ . As expected, the velocity components  $u_{ss}(y, t)$  and  $u_{cs}(y, t)$ , as well as those of the shear stresses, differ by a phase shift.

Finally, by making  $\lambda_4 = 0, \lambda_4 = \lambda_2 = 0, \lambda_4 = \lambda_3 = \lambda_2 = 0, \lambda_4 = \lambda_2 = \lambda_1 = 0$  or  $\lambda_4 = \lambda_3 = \lambda_2 = \lambda_1 = 0$  in eqs. (28) and (29), the steady-state solutions corresponding to Burgers, Oldroyd-B, Maxwell, second grade, respectively Newtonian fluids are obtained. As a check of general results, let us observe that for  $\lambda_1 = \lambda_2 = \lambda_4 = 0$ , eq. (28) reduces to the known solutions:

$$\tau_{ss}(y, t) = fe^{-By} \sin(\omega t - Ay) \quad \text{and} \quad \tau_{cs}(y, t) = fe^{-By} \cos(\omega t - Ay) \quad (30)$$

obtained from [30, eqs. (35) and (36)] by a different technique. Into relations (30):

$$A^2 = \left(\frac{\omega}{2\nu}\right) \frac{\sqrt{1 + \lambda_r^2 \omega^2} - \lambda_r \omega}{1 + \lambda_r^2 \omega^2}, \quad B^2 = \left(\frac{\omega}{2\nu}\right) \frac{\sqrt{1 + \lambda_r^2 \omega^2} + \lambda_r \omega}{1 + \lambda_r^2 \omega^2}, \quad \text{and} \quad \lambda_r = \lambda_3$$

Direct computations clearly show that  $\tau_{ss}(y, t)$  and  $\tau_{cs}(y, t)$ , given by eqs. (30), satisfy the governing eq. (8) with  $\lambda_1 = \lambda_2 = \lambda_4 = 0$  and  $\lambda_3 = \lambda_r$ . The similar solutions corresponding to Newtonian fluids [31, eqs. (21) and (23)] are immediately recovered making  $\lambda_r = 0$  into eqs. (30).

#### *Motion through a circular cylinder due to an oscillating longitudinal shear*

Consider an IGBF at rest in an infinite circular cylinder of radius  $R$ . After time  $t = 0^+$  the cylinder applies an oscillating longitudinal shear stress  $f \sin(\omega t)$  to the fluid. Due to the shear the fluid is gradually moved and its velocity is of the form (9)<sub>2</sub>. The governing equation for the shear stress  $\tau_v(r, t)$  is given by eq. (17) and the appropriate initial and boundary conditions are:

$$v(r, 0) = 0, \quad \tau_v(r, 0) = \frac{\partial \tau_v(r, 0)}{\partial t} = \frac{\partial^2 \tau_v(r, 0)}{\partial t^2} = 0 \quad \text{for} \quad r \in [0, R] \quad (31)$$

$$\tau_v(R, t) = f \sin(\omega t) \quad \text{for} \quad t \geq 0 \quad (32)$$

The shear stress distribution corresponding to this motion, as it results from [27, eq. (48) with  $W_2 = 0$  and  $W_1 = f$ ], is:

$$\begin{aligned} \tau_v(r, t) = & \frac{r}{R} f \sin(\omega t) + \frac{2\omega f}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} \sum_{i=1}^3 \frac{a_{in}}{\alpha_{in}^2 + \omega^2} [\omega \sin(\omega t) - \alpha_{in} \cos(\omega t)] + \\ & + \frac{2\omega f}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} \sum_{i=1}^3 \frac{a_{in} \alpha_{in}}{\alpha_{in}^2 + \omega^2} e^{\alpha_{in} t} \end{aligned} \quad (33)$$

where  $r_n$  are the positive roots of the transcendental equation  $J_1(Rr) = 0$ ,  $\alpha_{in} (i = 1, 2, 3)$  are the roots of the algebraic equation:

$$\lambda_2 s^3 + (\lambda_1 + \beta r_n^2) s^2 + (1 + \alpha r_n^2) s + \nu r_n^2 = 0, \quad a_{1n} = \frac{\alpha_{1n}^2 + \left(\frac{\lambda_1}{\lambda_2}\right) \alpha_{1n} + \left(\frac{1}{\lambda_2}\right)}{(\alpha_{3n} - \alpha_{1n})(\alpha_{2n} - \alpha_{1n})},$$

$$a_{2n} = \frac{\alpha_{2n}^2 + \left(\frac{\lambda_1}{\lambda_2}\right) \alpha_{2n} + \left(\frac{1}{\lambda_2}\right)}{(\alpha_{3n} - \alpha_{2n})(\alpha_{1n} - \alpha_{2n})}, \quad \text{and} \quad a_{3n} = \frac{\alpha_{3n}^2 + \left(\frac{\lambda_1}{\lambda_2}\right) \alpha_{3n} + \left(\frac{1}{\lambda_2}\right)}{(\alpha_{1n} - \alpha_{3n})(\alpha_{1n} - \alpha_{2n})} \quad \text{if} \quad \lambda_2 \neq 0$$

If  $\lambda_2 = 0$ , the solution of the problem is [27, eq. (49)]:

$$\tau_v(r, t) = \frac{r}{R} f \sin(\omega t) + \frac{2\omega f}{R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_0(Rr_n)} \begin{cases} b(r_n, t) & \text{for } \lambda_1 = \lambda_4 = 0 \\ c(r_n, t) & \text{for } \lambda_1 \text{ or } \lambda_4 \neq 0 \end{cases} \quad (34)$$

where  $b(r_n, t) = \frac{\omega(1 + \alpha r_n^2) \sin(\omega t) + \nu r_n^2 \cos(\omega t)}{(\nu r_n^2)^2 + \omega^2(1 + \alpha r_n^2)^2} - \frac{\nu r_n^2}{(\nu r_n^2)^2 + \omega^2(1 + \alpha r_n^2)^2} \exp\left(-\frac{\nu r_n^2 t}{1 + \alpha r_n^2}\right)$

$c(r_n, t) = \sum_{i=1}^2 d_i \frac{\omega \sin(\omega t) - \beta_i \cos(\omega t)}{\beta_i^2 + \omega^2} + \sum_{i=1}^2 \frac{d_i \beta_i}{\beta_i^2 + \omega^2} e^{\beta_i t}$ , where  $d_1 = \frac{1 + \lambda_1 \beta_1}{\beta_1 - \beta_2}$ ,  $d_2 = \frac{1 + \lambda_1 \beta_2}{\beta_2 - \beta_1}$ , and  $\beta_i (i = 1, 2)$  are the roots of the equation  $(\lambda_1 + \beta r_n^2)s^2 + (1 + \alpha r_n^2)s + \nu r_n^2 = 0$ .

In both cases the general solution for the shear stress is presented as a sum of steady-state and transient solutions. For large values of  $t$  the starting solutions tend to the steady-state solutions that are periodic in time and independent of the initial conditions. As it results from [27, eq. (56)] the steady-state solution can also be presented in the simple form:

$$\tau_v(r, t) = f \operatorname{Im} \left\{ \frac{I_1(r\sqrt{\gamma})}{I_1(R\sqrt{\gamma})} e^{i\omega t} \right\} \quad (35)$$

where  $\operatorname{Im}$  denotes the imaginary part of that which follows,  $I_1(\cdot)$  is a modified Bessel function, and

$$\gamma = \frac{\omega \lambda_3 \omega(1 - \lambda_2 \omega^2) - \lambda_1 \omega(1 - \lambda_4 \omega^2) + i[(1 - \lambda_2 \omega^2)(1 - \lambda_4 \omega^2) + \lambda_1 \lambda_3 \omega^2]}{\nu(1 - \lambda_4 \omega^2)^2 + \lambda_3^2 \omega^2} \quad \text{and} \quad i = \sqrt{-1}$$

The corresponding velocities can be determined in the same way as previously.

**Motion of a Burgers fluid through a circular cylinder that applies a constantly accelerating shear stress to the fluid**

Let us consider the motion of a Burgers fluid [ $\lambda_4 = 0$  into eq. (1)] through an infinite circular cylinder that, after time  $t = 0^+$ , applies a constantly accelerating longitudinal shear stress to the fluid. The governing equations and the initial conditions are the same used in the previous section. The boundary condition is:

$$\tau_v(R, t) = ft \quad \text{for} \quad t \geq 0 \quad (36)$$

The shear stress distribution  $\tau_v(r, t)$  corresponding to this motion is (see [32, eq. (37)] with  $\alpha = \beta = 1$ ) and our relation (17)):

$$\begin{aligned} \tau_v(r, t) = & \frac{r}{R} ft - \frac{2f}{\lambda_2 R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_2(Rr_n)} \sum_{p=0}^{\infty} \left( \frac{-\nu r_n^2}{\lambda_2} \right)^p \sum_{q=1}^p \frac{p! \lambda_3^q}{q!(p-q)!} \sum_{k=0}^{p+1} \frac{(p+1)!}{k!(p-k+1)!} \\ & \cdot \left( \frac{\lambda_1}{\lambda_2} \right)^k \left\{ G_{2\ell, p+k+1} \left[ -\left( \frac{1}{\lambda_2} \right), t \right] + \lambda_1 G_{2, \ell+1, p+k+1} \left[ -\left( \frac{1}{\lambda_2} \right), t \right] + \right. \\ & \left. + \lambda_2 G_{2, \ell+2, p+k+1} \left[ -\left( \frac{1}{\lambda_2} \right), t \right] \right\} \end{aligned} \quad (37)$$

where  $\ell = q + k - p - 2$  and the generalized  $G_{a,b,c}(\cdot, t)$  function is defined by:

$$G_{a,b,c}(d, t) = \sum_{j=0}^{\infty} \frac{d^j \Gamma(c+j)}{\Gamma(c)\Gamma(j+1)} \frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}$$

The solution corresponding to  $\lambda_2 = 0$ , namely [32, eq. (46)]

$$\tau_v(r, t) = \frac{r}{R} ft - \frac{2f}{\lambda_1 R} \sum_{n=1}^{\infty} \frac{J_1(rr_n)}{r_n J_2(Rr_n)} \sum_{p=0}^{\infty} \left( -\frac{vr_n^2}{\lambda_2} \right)^p \sum_{q=1}^p \frac{p! \lambda_3^q}{q!(p-q)!} \cdot \left\{ G_{1,q-p-2,p+1} \left[ -\left( \frac{1}{\lambda_1} \right) t \right] + \lambda_1 G_{1,q-p-1,p+1} \left[ -\left( \frac{1}{\lambda_1} \right) t \right] \right\} \quad (38)$$

gives the shear stress corresponding to the same motion of an Oldroyd-B fluid. The similar solutions for Maxwell or second grade fluids are obtained by making  $\lambda_3 = 0$  or  $\lambda_1 \rightarrow 0$ , respectively. The corresponding velocities are immediately obtained using eq. (12)<sub>2</sub> and the known identity:

$$\int_0^t G_{a,b,c}(d, s) ds = G_{a,b-1,c}(d, t)$$

### Steady motions induced by a circular cylinder that applies an oscillating couple to the fluid

All previous solutions have been developed using known solutions from the literature. In this section, based on the governing eq. (16), we establish exact steady-state solutions for the motion of an IGBF due to an infinite circular cylinder that applies an oscillating torque per unit length  $2\pi Rf \sin(\omega t)$  or  $2\pi Rf \cos(\omega t)$  to the fluid. The velocity of the fluid is of the form (9)<sub>1</sub> and the boundary condition is:

$$\tau_w(R, t) = f \sin(\omega t) \quad \text{or} \quad \tau_w(R, t) = f \cos(\omega t) \quad (39)$$

Let us denote by  $\tau_{ws}(r, t)$  and  $\tau_{wc}(r, t)$  the solutions corresponding to the two problems and by:

$$\tau(r, t) = \tau_{wc}(r, t) + i\tau_{ws}(r, t) \quad (40)$$

the complex shear stress that has to satisfy eq. (16) and the boundary condition:

$$\tau(R, t) = f e^{i\omega t} \quad (41)$$

Following Rajagopal and Bhatnagar [33] we seek a separable solution:

$$\tau(r, t) = S(r)T(t) \quad \text{with} \quad T'(t) = \lambda T(t) \quad (42)$$

Substituting eq. (42) into eq. (16), we obtain:

$$v(1 + \lambda\lambda_3 + \lambda^2\lambda_4) \left[ S'' + \left( \frac{1}{r} \right) S' - \left( \frac{4}{r^2} \right) S \right] = \lambda(1 + \lambda\lambda_1 + \lambda^2\lambda_2) S \quad (43)$$

where  $\lambda$  is the separation constant. On defining  $s = r \{ [\lambda(1 + \lambda\lambda_1 + \lambda^2\lambda_2)] / [v(1 + \lambda\lambda_3 + \lambda^2\lambda_4)] \}^{1/2}$  eq. (43) takes the form of a modified Bessel equation, namely [34, eq. (A-2.28)]:

$$s^2 \frac{d^2 S}{ds^2} + s \frac{dS}{ds} - (4 + s^2) S = 0 \quad (44)$$

#### Case of the motion through a cylinder

Solving eq. (44) and bearing in mind the boundary condition (41) we find for the shear stresses  $\tau_{wc}(r, t)$  and  $\tau_{ws}(r, t)$  the expressions:

$$\tau_{wc}(r, t) = f \operatorname{Re} \left\{ \frac{I_2(r\sqrt{\gamma})}{I_2(R\sqrt{\gamma})} e^{i\omega t} \right\}, \quad \tau_{ws}(r, t) = f \operatorname{Im} \left\{ \frac{I_2(r\sqrt{\gamma})}{I_2(R\sqrt{\gamma})} e^{i\omega t} \right\} \quad (45)$$

where Re and Im denote the real and imaginary parts, respectively, of that which follows,  $I_2(\cdot)$  is a modified Bessel function of the first kind and the imaginary constant  $\gamma = i\omega(1 + i\omega\lambda_1 - \omega^2\lambda_2)/[v(1 + i\omega\lambda_3 - \omega^2\lambda_4)]$  is the same with that defined in the section *Motion through a circular cylinder due to an oscillating longitudinal shear*.

The corresponding steady-state velocities, namely:

$$w_c(r, t) = \frac{f}{\rho\omega} \operatorname{Re} \left\{ \frac{\sqrt{\gamma} I_1(r\sqrt{\gamma})}{i I_2(R\sqrt{\gamma})} e^{i\omega t} \right\}, \quad w_s(r, t) = \frac{f}{\rho\omega} \operatorname{Im} \left\{ \frac{\sqrt{\gamma} I_1(r\sqrt{\gamma})}{i I_2(R\sqrt{\gamma})} e^{i\omega t} \right\} \quad (46)$$

are easily obtained using eq. (12)<sub>1</sub> and the identities (see [35, eq. (7.7)])

$$J_1(x) = iI_1(-ix), \quad J_2(x) = -I_2(-ix), \quad xI_2'(x) + 2I_2(x) = xI_1(x)$$

Direct computations clearly show that the solutions (45) and (46) satisfy both the governing eq. (10) and the boundary conditions (39). Finally, it is important to point out that the similar solutions for Burgers, Oldroyd-B, Maxwell, second grade, and Newtonian fluids can be again obtained as limiting cases of general solutions. The corresponding solutions for Newtonian fluids corresponding to the cosine oscillations of the shear, for instance, are:

$$w_{cN}(r, t) = \frac{f}{\mu} \sqrt{\frac{2\nu}{\omega}} \operatorname{Re} \left\{ \frac{I_1 \left[ (1+i)r\sqrt{\frac{\omega}{2\nu}} \right]}{(1+i)I_2 \left[ (1+i)R\sqrt{\frac{\omega}{2\nu}} \right]} e^{i\omega t} \right\}, \quad (47)$$

$$\tau_{wcN}(r, t) = f \operatorname{Re} \left\{ \frac{I_2 \left[ (1+i)r\sqrt{\frac{\omega}{2\nu}} \right]}{I_2 \left[ (1+i)R\sqrt{\frac{\omega}{2\nu}} \right]} e^{i\omega t} \right\}$$

#### Case of the motion around a cylinder

Stokes [36] established an exact steady-state solution for rotational oscillations of an infinite rod immersed in a Newtonian fluid. This solution has been extended to second grade and Oldroyd-B fluids by Rajagopal [37, eq. (17)] and Rajagopal and Bhatnagar [33, eq. (34)]. Here, we present the solutions corresponding to the motion of an IGBF around an infinite cylindrical rod that applies rotational oscillating shears of the form (39) to the fluid. The stress distributions corresponding to this problem, valid for  $r > R$ , are given by:

$$\tau_{wc}(r, t) = f \operatorname{Re} \left\{ \frac{K_2(r\sqrt{\gamma})}{K_2(R\sqrt{\gamma})} e^{i\omega t} \right\}, \quad \tau_{ws}(r, t) = f \operatorname{Im} \left\{ \frac{K_2(r\sqrt{\gamma})}{K_2(R\sqrt{\gamma})} e^{i\omega t} \right\} \quad (48)$$

where  $K_2(\cdot)$  is the modified Bessel function of the second kind. The corresponding steady-state velocities:

$$w_c(r, t) = \frac{f}{\rho\omega} \operatorname{Re} \left\{ \frac{i\sqrt{\gamma} K_1(r\sqrt{\gamma})}{K_2(R\sqrt{\gamma})} e^{i\omega t} \right\}, \quad w_s(r, t) = \frac{f}{\rho\omega} \operatorname{Im} \left\{ \frac{i\sqrt{\gamma} K_1(r\sqrt{\gamma})}{K_2(R\sqrt{\gamma})} e^{i\omega t} \right\} \quad (49)$$

are again obtained using eq. (12)<sub>1</sub> and the identity  $xK_2'(x) + 2K_2(x) = -xK_1(x)$ .

The Newtonian solutions corresponding to the sine oscillations of the shear, namely:

$$\begin{aligned} \tau_{wsN}(r, t) &= f \operatorname{Im} \left\{ \frac{K_2 \left[ (1+i)r\sqrt{\frac{\omega}{2\nu}} \right]}{K_2 \left[ (1+i)R\sqrt{\frac{\omega}{2\nu}} \right]} e^{i\omega t} \right\}, \\ w_{sN}(r, t) &= -\frac{f}{\mu} \sqrt{\frac{2\nu}{\omega}} \operatorname{Im} \left\{ \frac{K_1 \left[ (1+i)r\sqrt{\frac{\omega}{2\nu}} \right]}{(1+i)K_2 \left[ (1+i)R\sqrt{\frac{\omega}{2\nu}} \right]} e^{i\omega t} \right\} \end{aligned} \quad (50)$$

are obtained making  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$  into eqs. (48)<sub>2</sub> and (49)<sub>2</sub>.

### Conclusions

In the case of the motion over an infinite plate, the non-trivial shear stress  $\tau(y, t)$  satisfies a partial differential equation of the same form as the velocity  $u(y, t)$ . Based on this property, exact solutions for the motion of an IGBF produced by an infinite plate that applies a constant shear  $f$  to the fluid are developed using the velocity expression corresponding to the first problem of Stokes [14]. These solutions can be immediately particularized to give the similar solutions for Burgers fluids but they cannot be used for the other rate type fluids. This is the reason that we separately gave the shear stress corresponding to the same motion of Oldroyd-B fluids. As a check of this last result, the known solution for second grade fluids [9, eq. (4.1)<sub>2</sub>] has been obtained as a limiting case of general solutions. In the next, for completion, the exact steady-state solutions corresponding to the motion of an IGBF due to an infinite plate that applies oscillating shear stresses  $f\sin(\omega t)$  or  $f\cos(\omega t)$  to the fluid are presented in the simplest forms. These solutions, unlike the previous ones, can be particularized to give both the solutions for Burgers fluids and those for Oldroyd-B, Maxwell, second grade, and Newtonian fluids. The solutions for second grade and Newtonian fluids, as it was to be expected, are identical to those obtained in the literature by a different technique.

In the case of the motion due to an infinite circular cylinder, our results refer to the non-trivial shear stresses  $\tau_v(r, t)$  and  $\tau_w(r, t)$  corresponding to some axial or rotational motions of an IGBF. The shear stress corresponding to the axial motion satisfies the same partial differential equation as the rotational velocity  $w(r, t)$  while the shear stress for the rotational motions satisfies a different partial differential equation whose solution can easily be established. As direct applications of the first remark, the shear stress distributions corresponding to two different motions of an IGBF are developed using known results from the literature. They correspond to fluid motions due to longitudinal oscillating or constantly accelerating shear stresses on the boundary. In the first case, as expected, the starting solutions are presented as a sum of steady-state and transient solutions.

In the last section, based on the second remark concerning motions generated by an infinite circular cylinder, the steady-state solutions corresponding to the motion of an IGBF due to an oscillating couple on the boundary are determined. The first exact solutions, presented in terms of modified Bessel functions  $I_1(\cdot)$  and  $I_2(\cdot)$ , correspond to the motion through an infinite circular cylinder. The other solutions are presented in terms of the modified Bessel functions

$K_1(\cdot)$ ,  $K_2(\cdot)$  and correspond to the fluid motion around an infinite cylindrical rod. All solutions satisfy the boundary conditions and all governing equations and can be easily specialized to give the similar solutions for Burgers, Oldroyd-B, Maxwell, second grade, and Newtonian fluids. The Newtonian solutions (50), for instance, clearly satisfy eq. (15)<sub>1</sub> in view of the known identity  $xK_1'(x) - K_1(x) = -xK_2(x)$ .

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