

## MIXED CONVECTION BOUNDARY-LAYER FLOW OF A MICRO POLAR FLUID TOWARDS A HEATED SHRINKING SHEET BY HOMOTOPY ANALYSIS METHOD

by

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*A comprehensive study of 2-D stagnation flow of an incompressible micro polar fluid with heat transfer characteristics towards a heated shrinking sheet is analytically analyzed. The main goal of this paper, is to find the analytic solutions using a powerful technique namely the homotopy analysis method for the velocity and the temperature distributions and to study the steady mixed convection in 2-D stagnation flows of a micro polar fluid around a vertical shrinking sheet. The governing equations of motion together with the associated boundary conditions are first reduced to a set of self-similar non-linear ordinary differential equations using a similarity transformation and are then solved by the homotopy analysis method. Some important features of the flow and heat transfer for the different values of the governing parameters are analyzed, discussed, and presented through tables and graphs. The heat transfer from the sheet to the fluid decreases with an increase in the shrinking rate. Micro polar fluids exhibit a reduction in shear stresses and heat transfer rate as compared to Newtonian fluids, which may be beneficial in flow and thermal control of polymeric processing.*

Key words: *stagnation flow, heat transfer, micro polar fluid, shrinking sheet*

### Introduction

Stagnation point flows have applications in blood flow problems, the aerodynamics extrusion of plastic sheets, boundary-layer along material handling conveyers, the cooling of an infinite metallic plate in a cooling bath, and textile and paper industries.

Flows over the tips of rockets, aircrafts, submarines, and oil ships are some instances of stagnation flow applications [1]. Hiemenz [2] started the study of stagnation flow problem

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and reduced the Navier-Stokes equations for the forced convection problem to an ordinary differential equation of third order via a similarity transformation. Chamkha [3] solved the problem of the laminar steady viscous flow near a stagnation point with heat generating/absorbing using an implicit finite difference scheme. The steady 2-D flow over a semi infinite permeable flat surface with mass and heat transfer at the wall was discussed by Chamkha and Issa [4]. The steady 2-D stagnation point flow of a power law fluid over a stretched surface was studied by Mahapatra and Gupta [5]. The hydromagnetic stagnation point flow with heat transfer over a permeable surface was examined by Attia [6]. Nazar *et al.* [7] presented the numerical solution of unsteady boundary-layer flow of an incompressible viscous fluid in the stagnation point region over a stretching sheet using Keller box method. The problem of steady 2-D laminar MHD mixed convection stagnation point flow with mass transfer over a heated permeable surface was examined by Abdelkhalek [8] by using perturbation technique. Wang [9] investigated axisymmetric stagnation slip flow and heat transfer on a moving plate. The 2-D boundary-layer stagnation point flow over a stretching sheet in case of injection/suction through porous medium with heat transfer was presented numerically by Layek *et al.* [10]. The non-orthogonal stagnation flow and the heat transfer to a horizontal plate were investigated numerically by Paullet and Weidman [11]. The 2-D steady incompressible mixed convection non-orthogonal stagnation flow towards a heated or cooled stretching vertical plate was considered by Yian *et al.* [12]. Ishak *et al.* [13] presented the numerical study of steady stagnation point flow on a vertical surface through a porous medium using Keller box method. The solution of hydromagnetic steady laminar 2-D stagnation flow of a viscous incompressible electrically conducting fluid of variable thermal conductivity over a stretching sheet was obtained by Sharma and Singh [14] using shooting method. Stagnation point flow with convective heat transfer towards a shrinking sheet was investigated by Wang [15]. The steady 2-D MHD stagnation point flow of an upper convected Maxwell fluid over a stretching surface was analyzed by Hayat *et al.* [16] using homotopy analysis method (HAM). The unsteady free convection MHD boundary-layer at the stagnation point of a 2-D body and an axisymmetric body with prescribed surface heat flux or temperature was investigated by Takhar *et al.* [17]. The 2-D stagnation point flow in a porous medium was studied by Kumaran *et al.* [18] using perturbation technique. Kumaran *et al.* [19] studied the problem of MHD boundary-layer flow of an electrically conducting fluid over a stretching and permeable sheet with injection/suction through the sheet. The problem of unsteady mixed convection boundary-layer flow near the stagnation point on a heated vertical plate embedded in a porous medium was presented by Hassanien and Al-Arabi [20] by using Keller box method. Hayat *et al.* [21] presented the MHD stagnation point flow and heat transfer through a porous space bounded by a permeable surface. The numerical study of steady 2-D laminar forced MHD Hiemenz flow over a flat plate in a porous medium was considered by Awang Kechil and Hashim [22] using implicit scheme method.

The Newtonian model is inadequate to describe completely some modern scientific, engineering, and industrial processes which are made up of materials possessing an internal structure. The non-Newtonian fluids flow problems play a special challenge to the researchers. These fluids play an important role in engineering, scientific, and industrial applications. The scope of non-Newtonian fluids has significantly increased mainly due to their connection with applied sciences. The governing equations of motion for non-Newtonian fluids are highly non-linear and complicated as compared to Newtonian fluids. Hoyt and Fabula [23] predicted experimentally that the fluids having polymeric additives, display a significant reduction of shear stress and polymeric concentration (see *e. g.* Eringen [24]). The deformation of such materials can be well explained by theory of micro polar fluids given by Eringen [25, 26]. Micro polar flu-

ids have important applications in colloidal fluids flow, blood flows, liquid crystals, lubricants and flow in capillaries, heat and mass exchangers, *etc.* The boundary-layer flow of an incompressible micro polar fluid over a semi infinite plate was analyzed by Ahmadi [27]. The extensive other applications of micro polar fluids can be seen in Ariman *et al.* [28, 29]. Guram and Smith [30] studied the stagnation flow of a micro polar fluid with strong and weak interaction. A numerical study of steady incompressible micro polar fluid in a 2-D stagnation point flow towards a stretching sheet was investigated by Nazar *et al.* [31]. The numerical solution of flow and heat transfer characteristics of mixed convection in a micro polar fluid along a vertical flat plate with conduction effects was presented by Chang [32]. The 2-D unsteady boundary-layer flow and heat transfer of a viscous incompressible electrically conducting non-Newtonian fluid in the stagnation region in the presence of magnetic field was studied numerically by Kumari *et al.* [33] using implicit finite difference scheme. The problem of 2-D non-orthogonal stagnation flow of a micro polar fluid on a flat plate was analyzed by Lok *et al.* [34]. Ayub *et al.* [35] presented the analytical solution of stagnation point flow of a viscoelastic fluid towards a stretching surface by using homotopy method. Ishak *et al.* [36] investigated steady stagnation flow towards a vertical surface immersed in a micro polar fluid. The reduced system of self similar equations was then solved numerically by a method based on finite difference scheme. Abel and Nandeppanavar [37] studied the exact solution of heat transfer and MHD boundary-layer flow of viscoelastic fluid through a stretching sheet with non-uniform heat source in the presence of an externally applied magnetic field. A similarity transformation was used to convert the non-linear boundary-layer equation into ordinary differential equation. Ashraf *et al.* [38] investigated numerically the problem of laminar flow of a micro polar fluid in a channel with parallel porous walls of different permeability using finite difference scheme. Ashraf *et al.* [39, 40] studied the flow of micropolar fluids between two infinite parallel disks using finite difference method. The main goal of the present paper, is to investigate the 2-D stagnation point flow and heat transfer of a steady viscous incompressible micro polar fluid towards a vertical shrinking sheet in presence of buoyancy forces.

Numerical difficulties additionally appear if a non-linear problem contains singularities or has multiple solutions perturbation techniques are based on the existence of small/large parameters, the so-called *perturbation* quantity. Unfortunately, many non-linear problems in chemical engineering transport phenomena do not feature such types of perturbation quantities. Some non-perturbative techniques however have been developed and these include the artificial small parameter method [41], the  $\delta$ -expansion method [42], and the Adomian decomposition method [43]. Distinct from perturbation techniques, these non-perturbative methods are independent of small parameters. However, both of the perturbation techniques and the non-perturbative methods themselves fail to yield a simple method for adjusting or controlling the convergence region and rate of a given approximate series. To circumvent this difficulty, Liao [44, 45] employed the basic ideas of the homotopy in topology to propose a general analytic method for non-linear problems, namely the HAM. Based on homotopy of topology, the validity of the HAM is *independent* of whether there exist small parameters in the considered (differential) equations. Therefore, the HAM can overcome the foregoing restrictions and limits of perturbation methods. The HAM also avoids discretization and physically unrealistic assumptions, and provides an efficient, robust numerical solution with high accuracy and minimum computation. Further, the HAM always yields a family of solution expressions in the auxiliary parameter,  $\hbar$ . The convergence region and rate of each solution may be determined conveniently by the auxiliary parameter,  $\hbar$ . Also HAM is a rather general methodology and incorporates the homotopy perturbation method, Adomian decomposition method as well as the  $\delta$ -expansion

method. In recent years HAM has been successfully employed in simulating many types of non-linear problems including non-linear diffusion and reaction in porous catalysts [46], chaotic dynamical systems [47], and magneto-rheological wire coating analysis [48]. Further applications of HAM in fluid dynamics include the squeezing flow between circular plates [49], steady 3-D film condensation on an inclined rotating disk [50], the Burger and regularized long wave equations [51], hydromagnetic channel flows [52], and viscoelastic fluid flow over a stretching sheet with two auxiliary parameters [53]. This diverse range of problems testifies to the validity, effectiveness, and flexibility of the HAM, which holds strong potential in non-linear engineering fluid flows.

In the present article we implement HAM to derive totally analytical solutions for stagnation point flow of a micro polar fluid towards a heated shrinking sheet. The HAM solutions are compared with shooting quadrature computations, demonstrating the exceptional accuracy of HAM in simulating engineering.

### Problem statement and mathematical formulation

Consider 2-D stagnation point flow of a micro polar fluid impinging normally on a heated shrinking sheet. The flow is assumed to be laminar, steady, viscous, and incompressible. The equations of continuity, motion and energy for 2-D steady viscous incompressible boundary-layer flow of a micro polar fluid, neglecting the body couple, can be written as follows (for more details see [24, 34]).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \left( \frac{\mu + \kappa}{\rho} \right) \frac{\partial^2 u}{\partial y^2} + \frac{\kappa}{\rho} \frac{\partial N}{\partial y} \pm g\beta(T - T_\infty) \quad (2)$$

$$\rho j \left( u \frac{\partial N}{\partial x} + v \frac{\partial N}{\partial y} \right) = \gamma \frac{\partial^2 N}{\partial y^2} - \kappa \left( 2N + \frac{\partial u}{\partial y} \right) \quad (3)$$

$$\rho c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \kappa_0 \frac{\partial^2 T}{\partial y^2} \quad (4)$$

where  $N$  is the component of the micro rotation vector normal to the  $x$ - $y$  plane,  $\rho$  – the density,  $\kappa$  – the vortex viscosity,  $g$  – the acceleration due to gravity,  $\beta$  – the coefficient of thermal expansion,  $\gamma$  – the spin gradient viscosity,  $j$  – the microinertia density,  $p$  – the pressure of the fluid,  $T$  – the temperature,  $\kappa_0$  – the constant thermal conductivity, and  $c_p$  – the specific heat capacity at constant pressure of the fluid. All physical quantities  $\rho$ ,  $\mu$ ,  $\kappa$ ,  $\gamma$ , and  $j$  are assumed to be constants. The last term of eq. (2) refers to the buoyancy force (the influence of the thermal buoyancy force on the flow field). In energy equation viscous dissipation is neglected.

The “+” sign represents buoyancy assisting and “-” sign corresponds to the buoyancy opposing flow regions. In such a flow field (a vertical, heated surface) the upper half of the flow field is being assisted and the lower half of the flow field is being opposed by the buoyancy force. The boundary conditions for the velocity, micro rotation and temperature fields for the present problem are:

$$\left. \begin{aligned} u(x,0) = bx, \quad v(x,0) = 0, \quad u(x,\infty) = U = ax \\ N(x,0) = 0, \quad N(x,\infty) = 0, \quad T(x,0) = T_0, \quad T(x,\infty) = T_\infty \end{aligned} \right\} \quad (5)$$

where  $b < 0$  for the shrinking sheet,  $T_0$  is temperature on the surface,  $T_\infty$  – the temperature of the fluid at infinity, and  $U$  – the free stream velocity of the fluid. In order to obtain the velocity, mi-

cro rotation, and temperature fields for the problem under consideration, we have to solve eqs. (1)-(4) subject to the boundary conditions given in eq. (5). For this we use following similarity transformations:

$$\left. \begin{aligned} \eta &= \sqrt{\frac{a}{\nu}} y, & p(x, \infty) &= p_0 - \frac{\rho a^2}{2} (x^2 + y^2) \\ u(x, y) &= axf'(\eta), & v(x, y) &= -\sqrt{av}f(\eta) \\ N(x, y) &= -a\sqrt{\frac{a}{\nu}} xg(\eta), & \theta(\eta) &= \frac{T - T_\infty}{T_0 - T_\infty} \end{aligned} \right\} \quad (6)$$

where  $\eta$  is the similarity parameter, and  $p_0$  – the stagnation pressure. Using eq. (6) in eq. (1) we note that equation of continuity eq. (1) is identically satisfied and therefore the velocity field is compatible with continuity equation and represents the possible fluid motion. By the use of eq. (6) in eqs. (2)-(4) and after simplification, we get:

$$f'^2 - ff'' = (1 + R)f''' - Rg' + 1 + \lambda\theta \quad (7)$$

$$Cg'' + RA(f'' - 2g) = f'g - fg' \quad (8)$$

$$\theta'' + \text{Pr} f\theta' = 0 \quad (9)$$

where  $R = \kappa/\mu$ ,  $A = \mu/\rho ja$ ,  $C = \gamma/\mu j$ ,  $\text{Pr} = \mu c_p/\kappa_0$ , and  $\lambda = \pm \text{Gr}_x/\text{Re}_x^2$  (“+” in assisting flow and “-” in opposing flow) are the vortex viscosity parameter, the micro inertia density parameter, the spin gradient viscosity parameter, the Prandtl number, and buoyancy parameter, respectively. Here  $\text{Gr}_x = g\beta(T_0 - T_\infty)x^3/\nu^2$  is the local Grashof number and  $\text{Re}_x = Ux/\nu$  is the local Reynolds number. Boundary conditions given in eq. (5) in view of eq. (6) can be written:

$$\left. \begin{aligned} f(0) &= 0, & f'(0) &= \frac{b}{a} = \alpha, & f'(\infty) &= 1 \\ \theta(0) &= 1, & \theta(\infty) &= 0, & g(0) &= 0, & g(\infty) &= 0 \end{aligned} \right\} \quad (10)$$

where the case  $\alpha = 0$  stands for Hiemenz flow towards a solid plate and the case  $\alpha > 0$  is for the stagnation point flow over a stretching sheet. In case of our problem of stagnation flow towards a shrinking sheet, we take  $\alpha < 0$ . We note that eqs. (7)-(9) for vanishing micro rotation and  $\kappa = 0$  reduce to those obtained by Wang [15] and Attia [6] in case of Newtonian fluids. This validates our model for micro polar fluid flow.

### Analytical method

In this section we apply the HAM to obtain approximate analytical solutions of the investigations of stagnation point flow of a micro polar fluid towards a heated shrinking sheet, eqs. (7)-(9). We start with initial approximation  $f_0(\eta) = \eta + (\alpha - 1)(e^{-\eta} - 1)$ ,  $g_0(\eta) = 0$ ,  $\theta_0(\eta) = e^{-\eta}$  and the linear operators,  $\mathcal{L}$ :

$$\mathcal{L}_f(f) = \frac{\partial^3 f}{\partial \eta^3} + \frac{\partial^2 f}{\partial \eta^2} \quad (11)$$

$$\mathcal{L}_g(g) = \frac{\partial^2 g}{\partial \eta^2} + \frac{\partial g}{\partial \eta} \quad (12)$$

$$\mathcal{L}_\theta(\theta) = \frac{\partial^2 \theta}{\partial \eta^2} + \frac{\partial \theta}{\partial \eta} \quad (13)$$

The linear operators satisfy the following properties:

$$\mathcal{L}_f(c_1 + c_2\eta + c_3e^{-\eta}) = 0 \quad (14)$$

$$\mathcal{L}_g(c_4 + c_5e^{-\eta}) = 0 \quad (15)$$

$$\mathcal{L}_\theta(c_6 + c_7e^{-\eta}) = 0 \quad (16)$$

furthermore, eqs. (7)-(9) suggest defining the non-linear operator  $\mathcal{N}$ :

$$\begin{aligned} \mathcal{N}_f[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta; p)] = & (1 + R) \frac{\partial^3 \hat{f}(\eta; p)}{\partial \eta^3} + \hat{f}(\eta; p) \frac{\partial^2 \hat{f}(\eta; p)}{\partial \eta^2} - \\ & - \left( \frac{\partial \hat{f}(\eta; p)}{\partial \eta} \right)^2 - R \frac{\partial \hat{g}(\eta; p)}{\partial \eta} + 1 + \lambda \hat{\theta}(\eta; p) \end{aligned} \quad (17)$$

$$\begin{aligned} \mathcal{N}_g[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta; p)] = & C \frac{\partial^2 \hat{g}(\eta; p)}{\partial \eta^2} + RA \left( \frac{\partial^2 \hat{f}(\eta; p)}{\partial \eta^2} - 2\hat{g}(\eta; p) \right) + \\ & + \hat{f}(\eta; p) \frac{\partial \hat{g}(\eta; p)}{\partial \eta} - \frac{\partial \hat{f}(\eta; p)}{\partial \eta} \hat{g}(\eta; p) \end{aligned} \quad (18)$$

$$\mathcal{N}_\theta[\hat{f}(\eta; p), \hat{g}(\eta; p), \hat{\theta}(\eta; p)] = \frac{\partial^2 \hat{\theta}(\eta; p)}{\partial \eta^2} + \text{Pr} \hat{f}(\eta; p) \frac{\partial \hat{\theta}(\eta; p)}{\partial \eta} \quad (19)$$

The auxiliary functions  $\mathcal{H}$  are introduced:

$$\mathcal{H}_f(\eta) = \mathcal{H}_g(\eta) = \mathcal{H}_\theta(\eta) = e^{-\eta} \quad (20)$$

The  $m^{\text{th}}$  order deformation equations, eqs. (21)-(23), can be solved by the symbolic software MATHEMATICA:

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \hbar \mathcal{H}_f(\eta) R_{f,m}(\eta) \quad (21)$$

$$\mathcal{L}_g[g_m(\eta) - \chi_m g_{m-1}(\eta)] = \hbar \mathcal{H}_g(\eta) R_{g,m}(\eta) \quad (22)$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \hbar \mathcal{H}_\theta(\eta) R_{\theta,m}(\eta) \quad (23)$$

where  $\hbar$  is the auxiliary non-zero parameter:

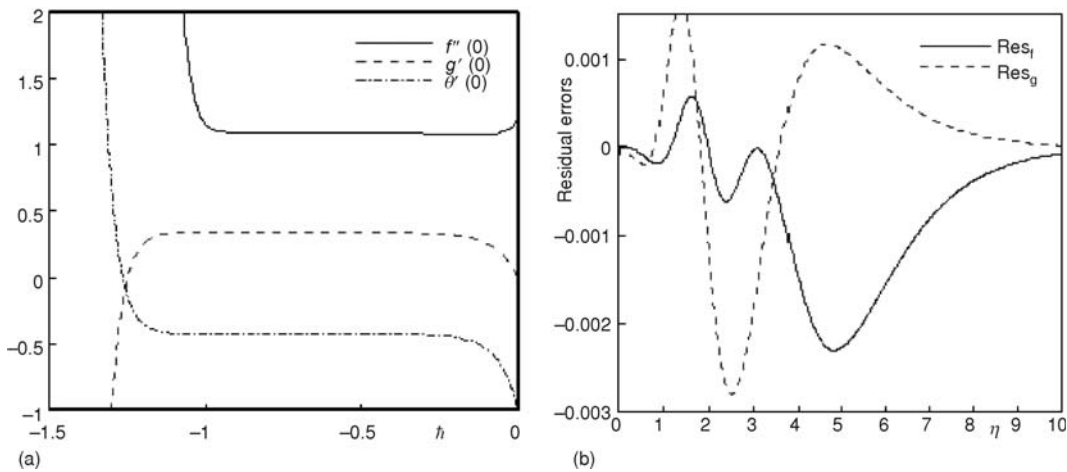
$$\begin{aligned} R_{f,m}(\eta) = & (1 + R) \frac{\partial^3 f_{m-1}(\eta)}{\partial \eta^3} - R \frac{\partial g_{m-1}(\eta)}{\partial \eta} + (1 - \chi_m) + \lambda \theta_{m-1}(\eta) + \\ & + \sum_{j=0}^{m-1} f_j(\eta) \frac{\partial^2 f_{m-1-j}(\eta)}{\partial \eta^2} - \sum_{j=0}^{m-1} \left( \frac{\partial f_j(\eta)}{\partial \eta} \frac{\partial f_{m-1-j}(\eta)}{\partial \eta} \right) \end{aligned} \quad (24)$$

$$\begin{aligned} R_{g,m}(\eta) = & C \frac{\partial^2 g_{m-1}(\eta)}{\partial \eta^2} + RA \left( \frac{\partial^2 f_{m-1}(\eta)}{\partial \eta^2} - 2g_{m-1}(\eta) \right) + \\ & + \sum_{j=0}^{m-1} f_j(\eta) \frac{\partial g_{m-1-j}(\eta)}{\partial \eta} - \sum_{j=0}^{m-1} g_j(\eta) \frac{\partial f_{m-1-j}(\eta)}{\partial \eta} \end{aligned} \quad (25)$$

$$R_{\theta,m}(\eta) = \frac{\partial^2 \theta_{m-1}(\eta)}{\partial \eta^2} + \text{Pr} \sum_{j=0}^{m-1} f_j(\eta) \frac{\partial \theta_{m-1-j}(\eta)}{\partial \eta} \quad (26)$$

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \quad (27)$$

For more information about the HAM solution and the other steps of using this method please see refs. [44, 45]. It is necessary to select a suitable value for auxiliary parameter to control and speed the convergence of the approximation series by the help of the so-called  $h$ -curve. The optimal values are selected from the valid region in straight line. In fig. 1(a) the  $h$ -curves are figured, obtained via 20<sup>th</sup> order of HAM solution. The averaged residual errors are defined as eqs. (28)-(30) to acquire optimal values of auxiliary parameters. In order to check the accuracy of the method, the residual errors of eqs. (28) and (29) are illustrated in fig. 1(b).



**Figure 1.** The obtained results by the HAM when  $R = 1$ ,  $A = 0.4$ ,  $C = 0.6$ ,  $\alpha = -0.2$ ,  $\text{Pr} = 0.7$ , and  $\lambda = 0.4$   
 (a)  $h$ -curves, (b) residual errors

$$\text{Res}_f = (1 + R) \left( \frac{d^3 f(\eta)}{d\eta^3} \right) - R \frac{dg(\eta)}{d\eta} - \left( \frac{df(\eta)}{d\eta} \right)^2 + f(\eta) \frac{d^2 f(\eta)}{d\eta^2} + \lambda \theta(\eta) + 1 \quad (28)$$

$$\text{Res}_g = C \frac{d^2 g(\eta)}{d\eta^2} + RA \left( \frac{d^2 f(\eta)}{d\eta^2} - 2g(\eta) \right) + f(\eta) \frac{dg(\eta)}{d\eta} - g(\eta) \frac{df(\eta)}{d\eta} \quad (29)$$

$$\text{Res}_\theta = \frac{d^2 \theta(\eta)}{d\eta^2} + \text{Pr} \left( f(\eta) \frac{d\theta(\eta)}{d\eta} \right) \quad (30)$$

### Results and discussion

This section is devoted for the presentation of our findings in tabular and graphical forms. In order to develop a better understanding of the physics of the problem, we choose to present the effects of shear and couple stresses, the shrinking parameter, micro polar parameters and the Prandtl number on the flow, and heat transfer characteristics. The values of micro polar parameters  $R$ ,  $A$ , and  $C$  given in tab. 1 are chosen arbitrarily in order to study their influence on the flow behavior (see Chang [32], Ashraf *et al.* [38-40], Guram and Anwar [54], and Takhar *et al.* [55]).

**Table 1. Values of  $R$ ,  $A$ , and  $C$  for the four cases discussed**

Case no.	$R$	$A$	$C$
1 (Newtonian)	0	0	0
2	1	0.2	0.3
3	3	0.4	0.5
4	5	0.6	0.7

**Table 2. Shear stresses, couple stresses, and heat transfer rate for  $R=1$ ,  $A=0.4$ ,  $C=0.6$ ,  $Pr=0.7$ ,  $\lambda=0$ , and various values of  $\alpha$** 

$-\alpha$	$f'(0)$	$g'(0)$	$\theta'(0)$
0.2	0.9412	0.3122	-0.4103
0.4	1.0002	0.3570	-0.3558
0.6	1.0173	0.3968	-0.2955
0.8	0.9788	0.4277	-0.2271
1	0.8538	0.4395	-0.1459

**Table 4. Heat transfer rate for  $\alpha=-0.2$ ,  $Pr=0.7$ ,  $\lambda=0$ , and four cases of values of  $R$ ,  $A$ , and  $C$** 

$Pr$	$-\theta'(0)$
0.1	-0.175916
0.4	-0.268102
0.7	-0.306968
1.0	-0.329594
2.0	-0.361808
3.0	-0.367916

wise velocity profiles fall with an increase in the values of  $R$ ,  $A$ , and  $C$  as shown in figs. 2(a) and (b). Furthermore, the velocity boundary-layer becomes thicker by increasing the values of the micro polar parameters  $R$ ,  $A$ , and  $C$  as shown in fig. 2(b). The micro rotation profiles rise with increasing values of  $R$ ,  $A$ , and  $C$  and there is no micro rotation for the Case 1 as this case corresponds to Newtonian fluids as predicted in fig. 2(c). Note that increasing the values of the parameters  $R$ ,  $A$ , and  $C$  has the effect of increasing the thermal boundary-layer thickness in the flow field as shown in fig. 2(d).

The effect of the shrinking parameter  $\alpha$  on the velocity and micro rotation fields for fixed values of the micro polar parameters and Prandtl number is predicted in fig. 3. Note that the normal velocity profiles  $f(\eta)$  fall with an increase in the magnitude of  $\alpha$  as shown in fig. 3(a). A reverse trend, caused due to shrinking of the sheet, occurs near the surface of the sheet and it increases as we increase the magnitude of shrinking parameter  $\alpha$ . Figure 3(b) predicts that the

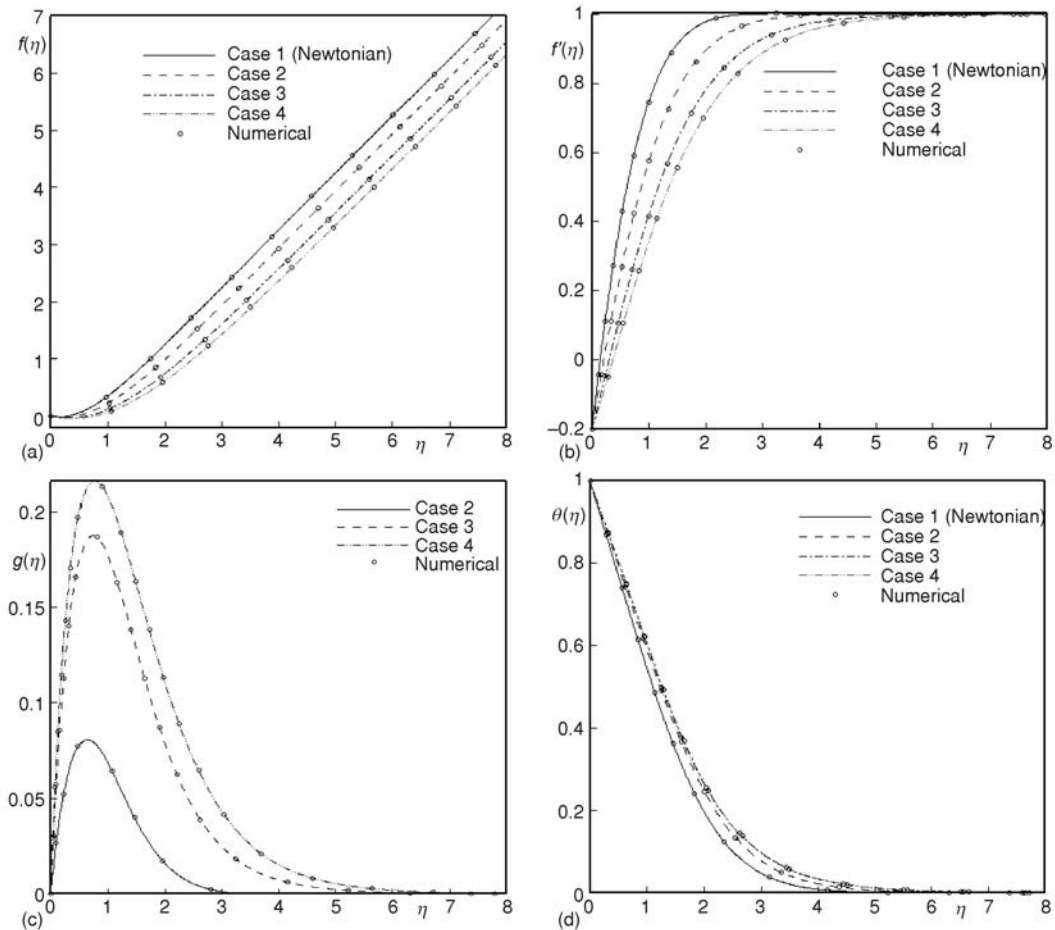
The values of shear stress  $f''(0)$  couple stress  $g'(0)$ , and heat transfer rate  $\theta'(0)$  at the sheet for various values of the shrinking parameter  $\alpha$  are given in tab. 2. The shear stresses increase for smaller values of magnitude of  $\alpha$ , however, a reverse trend can be observed for its larger values in magnitude in the given range  $0.2 \leq -\alpha \leq 0.1$ . The heat transfer from the sheet decreases, whereas the couple stresses increase, as the magnitude of  $\alpha$  is increased. Increase in the

**Table 3. Shear stresses, couple stresses, and heat transfer rate for  $\alpha=-0.2$ ,  $Pr=0.7$ ,  $\lambda=0$ , and four cases of values for  $R$ ,  $A$ , and  $C$** 

Case no.	$f''$	$g'(0)$	$\theta'(0)$
1 (Newtonian)	1.37397	0.000000	0.454348
2	0.948254	0.288399	0.410510
3	0.609857	0.621582	0.369700
4	0.466674	0.742934	0.347167

values of micro polar parameters  $R$ ,  $A$ , and  $C$  results in the reduction of shear stresses and heat transfer rate on the sheet while an opposite effect can be noted for couple stresses as shown in tab. 3. This is due to the fact that micro polar fluids offer a greater resistance to the fluid motion as compared to Newtonian fluids. This result, therefore, may be beneficial in flow and temperature control of polymeric processing. Note that the loss of heat transfer increases for various values of Prandtl number as shown in tab. 4.

In order to study the effects of governing parameters on the flow and heat transfer characteristics, we give graphical presentation of our analytical results. The effect of the parameters  $R$ ,  $A$ , and  $C$  on the flow and thermal fields for fixed values of  $\alpha$  and  $Pr$  is shown in fig. 2. The normal and stream



**Figure 2.** The obtained results by the HAM for different cases when  $\alpha = -0.2$ ,  $Pr = 0.7$ , and  $\lambda = 0.2$ ; (a) normal velocity profiles, (b) stream wise velocity profiles, (c) micro rotation profiles and (d) temperature profiles

stream wise velocity profiles  $f'(\eta)$  fall as the magnitude of shrinking rate increases for fixed values of  $R$ ,  $A$ ,  $C$ , and Prandtl number. The effect of  $\alpha$  on micro rotation is depicted in fig. 3(c). A rise in the micro rotation profiles is observed with an increase in the values of magnitude of  $\alpha$ . As one can obtain in assisting region,  $f(\eta)$  and  $f'(\eta)$  have higher values than opposing flow region and vice versa for micro rotation distribution,  $g(\eta)$  has higher values for opposing flow region.

The effect of  $\lambda$  on the flow and thermal fields for fixed values of other parameters is shown in fig. 4. The energy equation is coupled to momentum equation only with the buoyancy parameter. When  $\lambda$  increases, the velocity and the boundary-layer thickness increases (buoyancy assisting flow), as shown in fig. 4. As the buoyancy parameter  $\lambda$  increases, for the assisting flow, the velocity increases. This is obviously clear from the fact that assisting buoyant flow acts like a favorable pressure gradient that enhances the fluid motion. The opposite trend occurs for opposing flow. Opposing buoyant flow leads to an adverse pressure gradient, which slows down the flow. Note that  $\lambda = 0$  (the flow field not affected by the thermal field) leads to non-buoyant

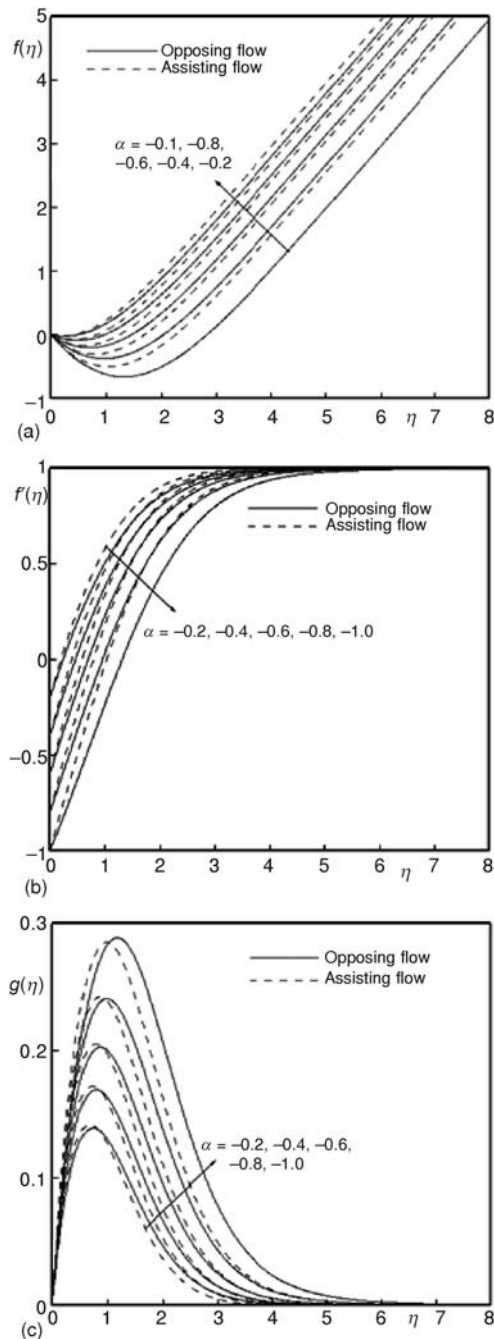


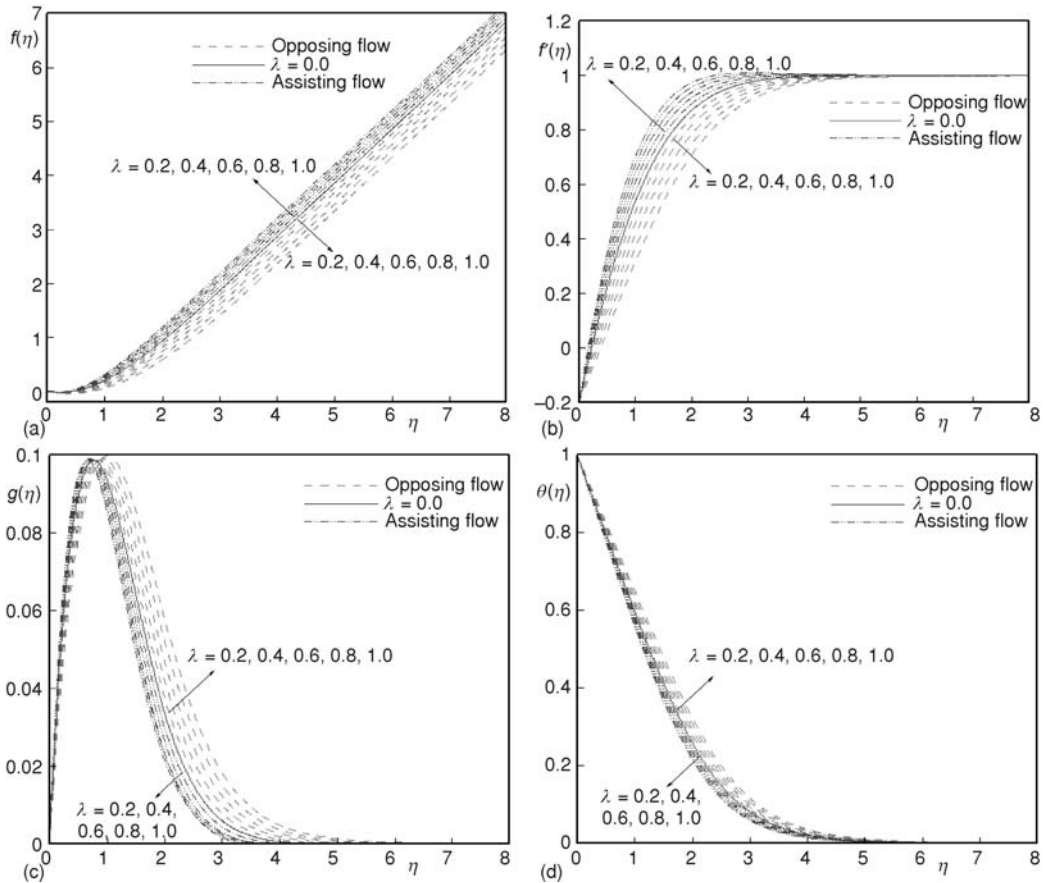
Figure 3. The obtained results by the HAM for different values of  $\alpha$  when  $R = 1$ ,  $A = 0.6$ ,  $C = 0.5$ ,  $Pr = 0.7$  and  $\lambda = \pm 0.2$ ; (a) normal velocity profiles, (b) stream wise velocity profiles and (c) micro rotation profiles

flow, which is called forced convection. Moreover, the thermal boundary-layer thickness decreases as  $\eta$  increases for both assisting and opposing flows. In assisting flow the temperature decreases with the increase in mixed convection parameter. On the other hand, in opposing flow region the temperature increases as figured in fig. 4(d).

### Conclusions

In this paper, the HAM is applied to study the stagnation point flow of a micro polar fluid towards a heated shrinking sheet in assisting and opposing flow regions. The validity of our solutions is verified by the numerical results. We analyzed the convergence of the obtained series solutions, carefully. Unlike perturbation methods, the HAM does not depend on any small physical parameters. Thus, it is valid for both weakly and strongly non-linear problems. Besides, the HAM provides us with a convenient way to control the convergence of approximation series, by means of auxiliary parameter  $\hbar$ , which is a fundamental qualitative difference in analysis between the HAM and other methods. It is apparently seen that the HAM is very powerful and efficient technique in finding analytical solutions for wide classes of non-linear partial differential equations. The solution obtained by means of the HAM is an infinite power series for appropriate initial approximation, which can be, in turn, expressed in a closed form.

This study considered the effects of shrinking parameter micro polar parameters and mixed convection parameter on 2-D laminar steady incompressible stagnation point flow and heat transfer of a micro polar fluid towards a shrinking sheet. A similarity transformation was used to convert the governing partial differential equations to ordinary ones. The transformed equations with associated boundary conditions were then solved analytically by HAM. The following conclusions can be made.



**Figure 4.** The obtained results by the HAM for different values of  $\lambda$  when  $R = 1$ ,  $A = 0.4$ ,  $C = 0.6$ ,  $Pr = 0.7$  and  $\alpha = -0.2$ ; (a) normal velocity profiles, (b) stream wise velocity profiles, (c) micro rotation profiles and (d) temperature profile

- A region of reverse flow occurs near the surface of the sheet due to the shrinking rate.
- The velocity and thermal boundary-layers become thicker with an increase in the values of shrinking parameter or micropolar parameters.
- The heat transfer rate from the sheet decreases with an increase in the magnitude of shrinking parameter and micropolar parameters.
- Micropolar fluids display a significant decrease in shear stress as compared to classical Newtonian fluids.

### Nomenclature

$x, y$  – Cartesian co-ordinates  
 $u, v$  – velocity components along x- and y-axes, respectively, [ $\text{ms}^{-1}$ ]

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