SOLUTION OF THE INVERSE HEAT CONDUCTION PROBLEM WITH NEUMANN BOUNDARY CONDITION BY USING THE HOMOTOPY PERTURBATION METHOD

by

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In the paper a solution of the inverse heat conduction problem with the Neumann boundary condition is presented. For finding this solution the homotopy perturbation method is applied. Investigated problem consists in calculation of the temperature distribution in considered domain, as well as in reconstruction of the functions describing the temperature and the heat flux on the boundary, in case when the temperature measurements in some points of the domain are known. An example confirming usefulness of the homotopy perturbation method for solving problems of this kind are also included.

Key words: *heat transfer, inverse problem, homotopy perturbation method*

Introduction

In many situations, both the physical and the technical ones, it is necessary to solve the so-called inverse problems. A common feature of such problems is that some information necessary for the full mathematical description of a considered model are either uncertain or missing. In the heat conduction problems the boundary conditions, some parameters or the geometry of body are the missing information.

The inverse problem solution is intended to retrieve the missing information and to determine the full description of the phenomenon under consideration (for example the temperature field). There are many of works in which the solution of inverse problem of the heat transfer was considered, for example [1-5]. Various, mainly numerical methods, were used there. In this type of problems, the heuristic approach, such as evolutionary algorithms, immunological algorithms or methods based on the swarm behavior were also successfully used and turned out to be effective [6-11]. In some cases a combined method was also applied. Such an approach can be found in works [12, 13] in which the sensitivity analysis was used. The sensitivity coefficients were found on analytical way while the direct problem was solved numerically.

Solving the inverse problems using analytical methods is not very popular, usually mainly because of the high complexity of the problem under consideration. There is only few works presenting such an approach [14-17]. The homotopy perturbation method presented in the paper is an example of the analytical method used for solving the inverse heat conduction problem.

Homotopy perturbation method was developed in the nineties of the last century by the Chinese mathematician He [18-20]. It is used, among others, to solve various problems con-

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nected with the heat transfer processes [21-23]. In papers [24, 25] this method was utilized for the inverse Stefan problem solution, whereas in work [26] this method was used for reconstruction of the missing boundary condition in the inverse heat conduction problem. For the inverse problem solutions, the homotopy perturbation method was also employed in works [27, 28]. Application of the discussed method for determining the temperature distribution in the cast-mould heterogeneous domain is presented in paper [29]. Shakeri and Dehghan [30] used this method to solution of the delay differential equation. Biazar and Ghazvini [31], in turn, applied the homotopy perturbation method for solving the hyperbolic partial differential equation. Application of the method for solving different kinds of differential equations can also be found in works [32-34]. Convergence of considered method in the case of differential equations is investigated in papers [35-37]. Homotopy perturbation method can also be applied for solving different kinds od integral equations [38-43].

In this paper the homotopy perturbation method is applied for solving the inverse heat conduction problem with the Neumann boundary condition. The problem consists in the calculation of temperature distribution in the domain, as well as in the reconstruction of functions describing the temperature and the heat flux on the boundary, in case when the temperature measurement in some points of the domain are known.

Problem formulation

In region $D = \{(x, t); x\in [0, b], t\in [0, t^*)\}$ we consider the heat conduction equation:

$$\frac{\partial u}{\partial t}(x,t) = a\,\frac{\partial^2 u(x,t)}{\partial x^2}, \quad (x,t)\in D \tag{1}$$

where a is the thermal diffusivity, u – the temperature, and t and x refer to time and spatial location, respectively. There are also given the initial condition:

$$u(x, 0) = \varphi(x), \quad x\in[0, b] \tag{2}$$

and the Neumann boundary condition:

$$-k\,\frac{\partial u(0,t)}{\partial x} = \eta(t), \quad t\in[0,t^*] \tag{3}$$

In the discussed inverse problem the temperature distribution u in the region D is determined as well as temperature θ and heat flux q on boundary for $x = b$, which define the Dirichlet and the Neumann boundary conditions:

$$u(b,t) = \theta(t), \quad t\in[0,t^*) \tag{4}$$

$$-k\,\frac{\partial u(b,t)}{\partial x} = q(t), \quad t\in[0,t^*) \tag{5}$$

The incomplete mathematical description is supplemented by temperature values at some fixed point $x = x_{\rm p}$, where $x_{\rm p}\in(0, b)$:

$$u(x_p,t) = \psi_p(t), \quad t\in[0,t^*) \tag{6}$$

Homotopy perturbation method

In the first step of method, the so-called homotopy operator is defined (for more details see for example [20, 24-26, 35]), which for the considered heat transfer equation takes the form

$$H(v,p)=\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u_0}{\partial x^2}+p\left(\frac{\partial^2 u_0}{\partial x^2}-\frac{1}{a}\frac{\partial v}{\partial t}\right) \tag{7}$$

where $p\in[0, 1]$ is the so-called homotopy parameter, $v(z,p) : D \times [0,1] \rightarrow \mathrm{R}$, and u_0 – the initial approximation of a solution of eq. (1). Because:

$$H(v,0)=\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u_0}{\partial x^2}$$

that for $p = 0$ the solution of operator equation $H(v,0) = 0$ is equivalent to solution of a trivial problem $(\partial^2 v/\partial x^2) - (\partial^2 u_0/\partial x^2) = 0$. For $p = 1$ the solution of operator equation $H(v, 1) = 0$ is equivalent to solution of the initial equation, since:

$$H(v,1)=\frac{\partial^2 v}{\partial x^2}-\frac{1}{a}\frac{\partial v}{\partial t}$$

Thus, changing the parameter p between 0 and 1 means changing the equation between trivial and given one (*i. e.* the solution v from u_0 to u).

Next, the solution of equation $H(v, p) = 0$ is searched in the form of power series:

$$v=\sum_{j=0}^{\infty} p^j v_j \tag{8}$$

If the above series has the radius of convergence not smaller than one, then by substituting p =1 the solution of considered equation is obtained:

$$u=\lim_{p\to 1} v=\sum_{j=0}^{\infty} v_j \tag{9}$$

Convergence of the considered series in case of the differential equations was studied in works [35-37], while in the integral equations case the convergence was considered in works [41-43].

In many cases this series is rapidly convergent, therefore the sum reduced to a few initial components provides a very good approximation of the desired solution. If we limit the sum to the first n + 1components, we obtain the so-called n-order approximate solution:

$$\hat{u}_n=\sum_{j=0}^{n} v_j \tag{10}$$

In order to find the function v_j, relation (8) is put into the equation $H(v, p) = 0$ which finally leads to:

$$\sum_{j=0}^{\infty} p^j \frac{\partial^2 v_j}{\partial x^2}=\frac{\partial^2 u_0}{\partial x^2}-p\frac{\partial^2 u_0}{\partial x^2}+\frac{1}{a}\sum_{j=1}^{\infty} p^j \frac{\partial v_{j-1}}{\partial t} \tag{11}$$

Comparison of the expressions with the same powers of parameter p gives the following equations:

$$v_0=u_0 \tag{12}$$

$$\frac{\partial^2 v_1}{\partial x^2}=\frac{1}{a}\frac{\partial v_0}{\partial t}-\frac{\partial^2 u_0}{\partial x^2} \tag{13}$$

and

$$\frac{\partial^2 v_j}{\partial x^2}=\frac{1}{a}\frac{\partial v_{j-1}}{\partial t} \quad \text{for} \quad j\ge 2 \tag{14}$$

Partial differential eqs. (13) and (14) must be supplemented by conditions ensuring a uniqness of the solution. For eq. (13) we assume the conditions:

$$\begin{cases} -k\dfrac{\partial v_0}{\partial x}(0,t) - k\dfrac{\partial v_1}{\partial x}(0,t) = \eta(t) \\ v_0(x_p,t) + v_1(x_p,t) = \psi_p(t) \end{cases} \tag{15}$$

while for eqs. (14) the conditions are in the form ($j \geq 2$):

$$\begin{cases} \dfrac{\partial v_j}{\partial x}(0,t) = 0 \\ v_j(x_p,t) = 0 \end{cases} \tag{16}$$

In this way, the solution of given problem was brought to the sequence of easy to solve partial differential equations. Looking for the solution of formulated problem, we need to define yet an initial approximation u_0, which can be assumed as the function determining the initial condition:

$$u_0(x, t) = \varphi(x) \tag{17}$$

Knowing the exact temperature distribution $u(x, t)$ or its approximation $\hat{u}_n(x, t)$, which is defined by differentiable functions, the missing boundary conditions could be determined: $\theta(t) = u(b, t)$, $q(t) = -k[\partial u(b, t)/\partial x]$, or adequately: $\theta(t) = \hat{u}_n(b, t)$, $q(t)] = -k[\partial \hat{u}_n(b, t)/\partial x]$.

Example

Application of the method will be illustrated by the example in which we assume: $b = 2$, $x_p = 0.75$, $a = 1/10$, $k = 2$, $t^* = 2$ and $\varphi(x) = e^{1-x}$, $\eta(t) = 2e^{t/10+1}$, and $\psi_p(t) = e^{t/10 + 1/4}$.

Exact solution of the problem formulated is defined by functions $u(x, t) = e^{1-x+t/10}$, $\theta(t) = e^{t/10-1}$, and $q(t) = 2e^{t/10-1}$.

As the initial approximation u_0 we assume the function satisfying the initial condition $u_0(x, t) = e^{1-x}$.

Solving now eq. (13) with boundary condition (15) we determine $v_1(x, t) = e^{1/4 + t/10} - e^{1-x} + e^{1+t/10}[(3/4) - x]$.

Again, the functions v_j, $j \geq 2$, are determined recursively by solving eq. (14) with boundary conditions (16). For example we obtain:

$$v_2(x,t) = \frac{1}{192}e^{\frac{1}{4}+\frac{t}{10}}[54 - 96x^2 + e^{3/4}(27 - 72x^2 + 32x^3)]$$

$$v_3(x,t) = \frac{-1}{30720}e^{\frac{1}{4}+\frac{t}{10}}(4x-3)\cdot[675 + 900x - 240x^2 - 320x^3 + 4e^{3/4}\cdot(9 + 6x - 4x^2)^2]$$

$$v_4(x,t) = \frac{1}{20643840}e^{t/10}[7e^{1/4}\cdot(-44469 + 97200x^2 - 34560x^4 + 4096x^6) - 4e(37179 - 81648x^2 + 30240x^4 - 5376x^6 + 1024x^7)]$$

In tab. 1 the errors of retrieved function describing the temperature distribution in the considered area are presented. Table 2 con-

Table 1. Errors of the reconstructed temperature distribution (Δ_u – absolute error, δ_u relative eerror)

n	Δ_u	δ_u [%]
1	1.22532	82.05888
2	0.10319	6.91049
3	2.76294310^{-2}	1.85031
4	$5.92860\cdot10^{-3}$	0.39703
5	$1.35805\cdot10^{-3}$	$9.09477\cdot10^{-2}$
6	$3.09511\cdot10^{-4}$	$2.07276\cdot10^{-2}$
7	$7.05608\cdot10^{-5}$	$4.72539\cdot10^{-3}$
8	$1.60859\cdot10^{-5}$	$1.07726\cdot10^{-3}$
9	$3.66613\cdot10^{-6}$	$2.45518\cdot10^{-4}$
10	$8.34587\cdot10^{-7}$	$5.58916\cdot10^{-5}$
11	$1.94339\cdot10^{-7}$	$1.30147\cdot10^{-5}$
12	$3.72286\cdot10^{-8}$	$2.49317\cdot10^{-6}$

Table 2. Errors of the reconstructed boundary condition (Δ – absolute error, δ – relative error)

n	Δ_θ	δ_θ [%]	Δ_q	δ_q [%]
1	2.75186	674.59772	5.21252	638.90561
2	0.22621	55.45269	2.53150	310.28982
3	$3.30981\cdot10^{-2}$	8.11377	$2.25002\cdot10^{-2}$	2.75789
4	$3.58087\cdot10^{-3}$	0.87782	$3.45919\cdot10^{-2}$	4.23998
5	$9.21347\cdot10^{-4}$	0.22586	$6.51227\cdot10^{-3}$	0.79822
6	$2.08268\cdot10^{-4}$	$5.10554\cdot10^{-2}$	$1.51334\cdot10^{-3}$	0.18549
7	$4.75000\cdot10^{-5}$	$1.16443\cdot10^{-3}$	$3.44600\cdot10^{-4}$	$4.22381\cdot10^{-2}$
8	$1.08285\cdot10^{-5}$	$2.65454\cdot10^{-3}$	$7.85633\cdot10^{-5}$	$9.62962\cdot10^{-3}$
9	$2.46861\cdot10^{-6}$	$6.05162\cdot10^{-4}$	$1.79103\cdot10^{-5}$	$2.19529\cdot10^{-3}$
10	$5.62776\cdot10^{-7}$	$1.37960\cdot10^{-4}$	$4.08305\cdot10^{-6}$	$5.00465\cdot10^{-4}$
11	$1.28297\cdot10^{-7}$	$3.14512\cdot10^{-5}$	$9.30823\cdot10^{-7}$	$1.14092\cdot10^{-4}$
12	$2.92483\cdot10^{-8}$	$7.17002\cdot10^{-6}$	$2.12202\cdot10^{-7}$	$2.60099\cdot10^{-5}$

tains the errors of reconstructed missing boundary conditions, *i. e.* functions θ and q, for various numbers of components n. Results placed in the paper show that the errors decrease rapidly with the increase of components number in sum (8). The graph of errors of boundary condition reconstruction are shown also in figs. 1 and 2. They present the approximations of order 7 and 12 of

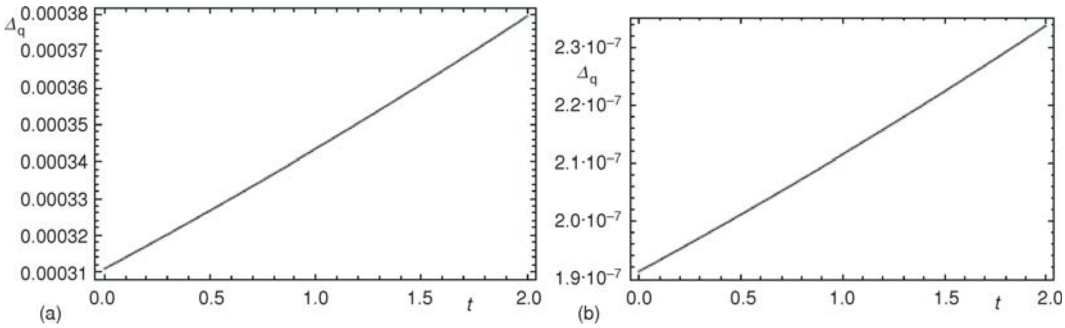

Figure 1. Errors of the reconstructed heat flux on the boundary (*a* – for *n* =7, *b* – for *n* = 12)

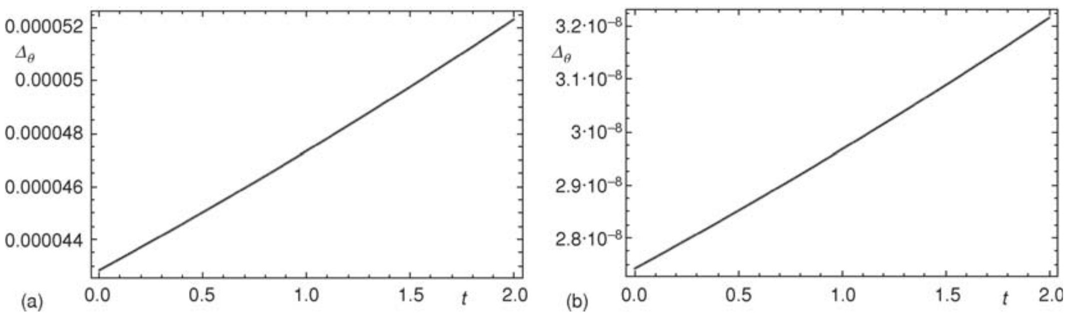

Figure 2. Errors of the reconstructed temperature on the boundary (*a* – for *n* = 7, *b* – for *n* = 12)

the heat flux. The initial condition is retrieved with small errors which for $n = 7$ were equal to $\Delta = 6.36339\cdot10^{-5}$, $\delta = 4.72539\cdot10^{-3}$% and for $n = 12$ were equal to $\Delta = 3.91834\cdot10^{-8}$, $\delta = 2.90972\cdot10^{-6}$%.

Calculations for the different positions of control point x_p (0.25, 0.5, and 1.0) have been also performed and the results obtained for 8th-order approximate solution are presented in tab. 3 and figs. 3 and 4. Every time the very good reconstruction of searched values was obtained. It can be noted that the position of control point affects for the rate of series convergence (8). Location closer to the left boundary of the considered region means that the determined series rapidly converges.

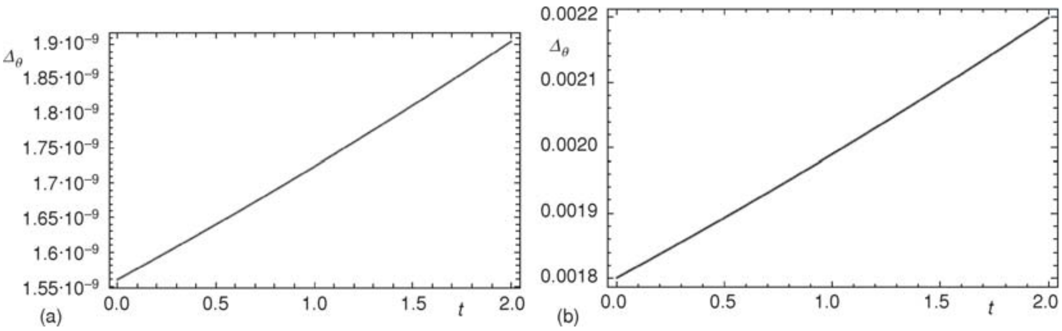

Figure 3. Errors of the reconstructed temperature on the boundary for 8th-order approximate solution and x_p = 0.25 (a) and x_p =1.0 (b)

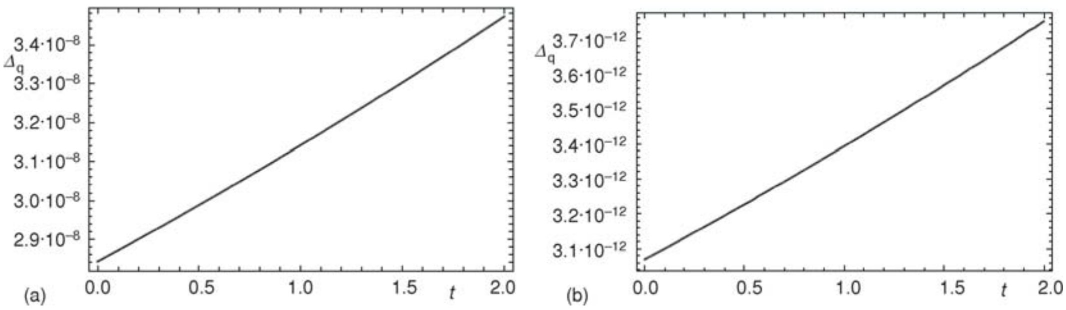

Figure 4. Errors of the reconstructed heat flux on the boundary for 8th-order approximate solution and x_p = 0.25 (a) and x_p = 1.0 (b)

Table 3. Errors of the reconstructed values for different localization of the control point x_p and for 8th-order approximate solution (Δ – absolute error, δ – relative error)

x_p	Δ_u	δ_u	Δ_θ	δ_θ	Δ_q	δ_q
0.25	$5.26492\cdot10^{-8}$	$3.52587\cdot10^{-6}$	1.72957^{-9}	4.23992^{-7}	3.15236^{-8}	$3.86389\cdot10^{-6}$
0.5	1.86143^{-8}	$1.24658\cdot10^{-6}$	3.59116^{-8}	8.80348^{-6}	2.46860^{-9}	$3.02580\cdot10^{-7}$
0.75	$1.60859\cdot10^{-5}$	$1.07726\cdot10^{-3}$	1.08285^{-5}	2.65454^{-3}	7.85633^{-5}	$9.62962\cdot10^{-3}$
1.0	$1.41199\cdot10^{-3}$	$9.45598\cdot10^{-2}$	$1.99686\cdot10^{-3}$	4.89515^{-1}	3.40371^{-12}	$4.17197\cdot10^{-10}$

Conclusions

In this work the application of the homotopy perturbation method for solving the inverse heat conduction problem with Neumann boundary condition has been presented. Problem consists in the calculation of temperature distribution in the domain, as well as in the reconstruction of functions describing the temperature and the heat flux on the boundary, in case when the temperature measurement in some points of the domain are known. Investigated method was tested on some examples (one of them is presented in the work) and results show that the accuracy of obtained solutions is very good. The reduction in the volume calculations and its rapid convergence show that the method is a powerful and straightforward tool in solving the considered problem.

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