

ANALYTICAL INVESTIGATION OF NON-LINEAR MODEL ARISING IN HEAT TRANSFER THROUGH THE POROUS FIN

by

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In this paper simple analytical methods called homotopy perturbation method, variation iteration method, and perturbation method are employed to approach temperature distribution of porous fins. Also energy balance and Darcy's model used to formulate the heat transfer equation. To study the thermal performance, a type case considered is finite-length fin with insulated tip. The obtained results from variation iteration method are compared with other analytical techniques proposed before. These methods are homotopy perturbation method and perturbation method. Also boundary value problem is applied as a numerical method for validation. The obtained results shows that the variational iteration method is more accurate, stable, and more appropriate than other techniques. Also it is found that this method is powerful mathematical tools and can be applied to a large class of linear and non-linear problems arising in different fields of science and engineering specially some heat transfer equations.

Key words: heat transfer, porous fin, homotopy perturbation method, variational iteration method, perturbation method

Introduction

The heat transfer rate enhancement in fins with reducing size and cost is the aim of many researchers in engineering applications [1-3]. To achieve this goals, convective heat transfer coefficient, surface area available, and temperature difference between surface and surrounding fluid are such as ways can be used.

Non-linear problems and phenomena play an important role in applied mathematics, physics, engineering, and other branches of science specially some heat transfer equations. Except for a limited number of these problems, most of them do not have precise analytical solutions; therefore, these non-linear equations should be solved using approximation methods. Perturbation method is one of the well-known methods to solve non-linear problems; it is based on the existence of small/large parameters, the so-called perturbation quantity [4, 5]. Many non-linear problems do not contain such kind of perturbation quantity, and we can use non-perturbation

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methods, such as the artificial small parameter method [6], the δ -expansion method [7], the Adomian's decomposition method [8], the homotopy perturbation method (HPM) [9-13], the variational iteration method (VIM) [14-30], the optimal homotopy asymptotic method (OHAM) [31-32] and the optimal homotopy perturbation method (OHPM) [33].

The present work, the basic idea of the homotopy perturbation method, variational iteration method, and perturbation method are introduced and then we have applied to find the approximate solutions of non-linear differential equations governing on porous fin. Result demonstrates the variational iteration method is simple and offers superior accuracy compared with the perturbation method and homotopy perturbation method. Also it is found that these method are powerful mathematical tools and that they can be applied to a large class of linear and non-linear problems arising in different fields of science and engineering.

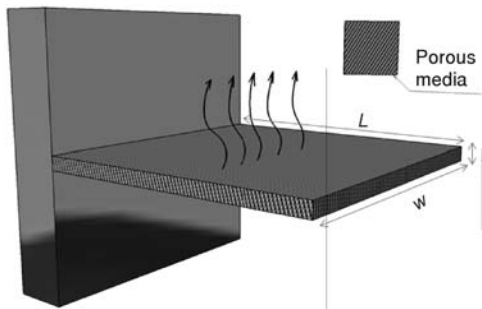


Figure 1. Schematic diagram of fin profile under consideration

There is no temperature variation across the fin thickness. The solid matrix and fluid are assumed to be at local thermal equilibrium with each other. The interactions between the porous medium and the clear fluid can be simulated by the Darcy formulation.

Apply an energy balance to the slice segment of the fin of thickness Δx shown in fig. 1, requires that:

$$q(x) - q(x + \Delta x) = \dot{m}c_p[T(x) - T_\infty] + h(p\Delta x)[T(x) - T_\infty] \quad (1)$$

The mass flow rate of the fluid passing through the porous material can be written as:

$$\dot{m} = \rho v_w \Delta x W \quad (2)$$

From the Darcy's model we have:

$$v_w = \frac{gk\beta}{\nu(T - T_\infty)} \quad (3)$$

Substitutions of eqs. (2) and (3) into eq. (1) yields:

$$\frac{q(x) - q(x + \Delta x)}{\Delta x} = \frac{\rho c_p g k \beta w}{\nu} [T(x) - T_\infty]^2 + hp[T(x) - T_\infty] \quad (4)$$

as, $\Delta x \rightarrow 0$ eq. (4) becomes:

$$\frac{dq}{dx} = \frac{\rho c_p g k \beta w}{\nu} [T(x) - T_\infty]^2 + hp[T(x) - T_\infty] \quad (5)$$

Governing equations

As shown in fig. 1, a rectangular fin profile is considered. The dimensions of the fin are length L , width w , and thickness t . The cross-section area of the fin is constant. This fin is porous to allow the flow of infiltrate through it. The following assumptions are made to solve this problem. The porous medium is isotropic and homogeneous. The porous medium is saturated with single-phase fluid, the surface radiant exchange is neglected. Physical properties of both fluid and solid matrix are constant. The temperature inside fin is only function of X .

from Fourier's Law of conduction, we have:

$$q = -k_{\text{eff}} A \frac{dT}{dx} \quad (6)$$

where A is the cross-sectional area of the fin $A = wt$ and k_{eff} – the effective thermal conductivity of the porous fin given by $k_{\text{eff}} = \phi k_f + (1 - \phi)k_s$. Substitution eqs. (6) into eq. (5) gives:

$$\frac{d^2 T}{dx^2} - \frac{\rho c_p g k \beta}{t k_{\text{eff}} v} [T(x) - T_\infty]^2 - \frac{hp}{k_{\text{eff}} A} [T(x) - T_\infty] \quad (7)$$

Hence, with applying energy balance equation at steady-state condition (see fig. 2), and introducing non-dimensional temperature function where, $\theta = [T(x) - T_\infty]/T_b - T_\infty$ and $X = x/L$ into eq. (7) we have:

$$\frac{d^2 \theta}{dX^2} - L^2 S_h \theta^2 - M^2 \theta = 0 \quad (8)$$

Porous parameter, $S_h = \text{DaRa}/k_f(L/t)^2$ and Convection parameter, $M^2 = (hp/k_s A)$ where S_h is a porous parameter that indicates the effect of the permeability of the porous medium as well as buoyancy effect so higher value of S_h indicate higher permeability of the porous medium or higher buoyancy forces. M is a convection parameter that indicates the effect of surface convection of the fin.

For finite-length fin with insulated tip:

$$\theta(1) = 1, \quad \theta'(0) = 0 \quad (9)$$

Variational iteration method

To illustrate the basic idea of VIM, we consider the following general non-linear system:

$$Lu + Nu = g(x) \quad (10)$$

where L is a linear operator, N – a non-linear operator, and $g(x)$ a homogeneous term. According to the VIM, we can construct the following iteration formulation:

$$u_{n+1}(t) = u_n(t) + \int_0^1 \lambda [Lu_n(\tau) + Nu_n(\tau) - g(\tau)] d\tau \quad (11)$$

where λ is a general Lagrangian multiplier [34, 35], which can be identified optimally via the variational theory [36, 37]. The subscript n indicates the n^{th} approximation and \tilde{u}_n is considered as a restricted variation [38, 39], i. e., $\delta \tilde{u}_n = 0$.

The application of VIM

First we construct a correction functional which reads:

$$\theta_{n+1}(x) = \theta_n(x) + \int_0^x \lambda \left[\frac{d^2 \theta_n(\tau)}{d\tau^2} - L^2 S_h \tilde{\theta}_n^2(\tau) - M^2 \theta_n(\tau) \right] d\tau \quad (12)$$

where λ is the general Lagrange multiplier.

Making the above correction functional stationary, we can obtain following stationary conditions:

$$\lambda''(t) - M^2 \lambda(t) = 0, \quad 1 - \lambda'(t)|_{t=x} = 0, \quad \lambda(t)|_{t=x} = 0 \quad (13)$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda = -\frac{1}{2} \left(\frac{e^{M(x-\tau)} - e^{M(\tau-x)}}{M} \right) \quad (14)$$

As a result, we obtain the following iteration formula:

$$\theta_{n+1}(X) = \theta_n(X) + \int_0^X -\frac{1}{2} \left(\frac{e^{M(x-\tau)} - e^{M(\tau-x)}}{M} \right) \left[\frac{d^2 \theta_n(\tau)}{d\tau^2} - L^2 S_h \tilde{\theta}_n^2(\tau) - M^2 \theta_n(\tau) \right] d\tau \quad (15)$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$\theta_0(x) = \operatorname{sech}(M) \cosh(Mx) \quad (16)$$

Using the variational eq. (15) for $n = 0$, substituting eqs. (16) into (15) and after some simplifications, we have:

$$\theta_1(x) = C_0 \left[\operatorname{sech}(M) \cosh(Mx) + \frac{0.083333L^2 S_h (2e^{3Mx} + 2e^{Mx} + e^{4Mx} + 1 - 6e^{2Mx}) e^{-2Mx}}{M^2 \cosh(M)^2} \right] \quad (17)$$

where $C_0 = 1/A$, that $A = \operatorname{sech}(M) \cosh(M) + [0.083333L^2 S_h (2e^{3M} + 2e^M + e^{4M} + 1 - 6e^{2M}) e^{-2M}] / M^2 \cosh(M)^2$ and so on. In the same way, the rest of the components of the iteration formula can be obtained.

Analysis of He's homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \quad (18)$$

with the boundary condition of:

$$B \left(u, \frac{\partial u}{\partial n} \right) = 0, \quad r \in \Gamma \quad (19)$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function, and Γ is the boundary of the domain Ω . A can be divided into two parts, which are L and N , where L is linear and N is non-linear. Equation (8) can therefore be re-written as:

$$L(u) + N(u) - f(r) = 0, \quad r \in \Omega \quad (20)$$

Homotopy perturbation structure is shown as:

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \quad (21)$$

where

$$v(t, p): \Omega \times [0, 1] \rightarrow R \quad (22)$$

In eq. (21), $p \in [0, 1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of eq. (8) can be written as a power series in p , as following:

$$v = v_0 + p v_1 + p^2 v_2 + \dots = \sum_{i=0}^n v_i p^i \quad (23)$$

and the best approximation for solution is:

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots \quad (24)$$

The application of HPM

In this section, we will apply the HPM to non-linear ordinary differential eq. (8) with a boundary condition (9). According to the HPM, we can construct homotopy of eq. (8) as follows:

$$(1 - P)[\theta''(x) - \theta(x)] + p[\theta''(x) - R\theta^2(x) - M^2\theta(x)] = 0 \quad (25)$$

where $R = L^2S_h$ is constant. We consider θ as follows:

$$\theta(x) = \theta_0(x) + P\theta_1(x) + P^2\theta_2(x) + \dots = \sum_{i=0}^n P^i\theta_i(x) \quad (26)$$

From eq. (23), if the two terms approximations are sufficient, we will obtain with substituting θ from eq. (26) into eq. (25) and some simplification and re-arranging based on powers of p -terms, with assumption $M = 1$ we have:

$$p^0: \begin{aligned} -\theta_0(x) + \theta_0''(x) &= 0 \\ \theta_0(1) &= 1, \theta_0'(0) = 0 \end{aligned} \quad (27)$$

$$p^1: \begin{aligned} \theta_1''(x) + \theta_1(x) - R\theta_0^2(x) &= 0 \\ \theta_1(1) &= 0, \theta_1'(0) = 0 \end{aligned} \quad (28)$$

Solving eqs. (27) and (28) with boundary conditions, we have:

$$\theta_0(x) = \frac{e^x}{e + e^{-1}} + \frac{e^{-x}}{e + e^{-1}} \quad (29)$$

$$\begin{aligned} \theta_1(x) &= \frac{1}{3} \frac{e^{-x}R(-1 + 6e^2 - e^4)}{ee^4 + 2ee^2 + e + 2e^{-1}e^2 + e^{-1}e^4 + e^{-1}} + \\ &+ \frac{1}{3} \frac{e^xR(-1 + 6e^2 - e^4)}{ee^4 + 2ee^2 + e + 2e^{-1}e^2 + e^{-1}e^4 + e^{-1}} - \frac{R(-1 + 6e^2 - e^4)e^{-2x+2}}{3 + 6e^2 + 3e^4} \end{aligned} \quad (30)$$

The solution of this equation, when $p \rightarrow 1$, will be:

$$\theta(x) = \theta_0(x) + \theta_1(x) \quad (31)$$

The application of perturbation method

For low values of S_h , we can use ε instead. Now for very small ε , let us assume a regular perturbation expansion and calculate the first three terms, thus we assume [5]:

$$\theta = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \dots \quad (32)$$

After substituting eqs. (32) into (8), collecting terms with the powers of ε as 0, 1, 2... and equating coefficients of each power of ε on both sides we have:

$$\varepsilon^0: \begin{aligned} \frac{d^2\theta_0(x)}{dx^2} - \theta_0(x) &= 0 \\ \theta_0(1) &= 1, \theta_0'(0) = 0 \end{aligned} \quad (33)$$

$$\varepsilon^1: \begin{aligned} \frac{d^2\theta_1(x)}{dx^2} - \theta_1(x) - [\theta_0(x)]^2 &= 0 \\ \theta_1(1) &= 1, \theta_1'(0) = 0 \end{aligned} \quad (34)$$

$$\varepsilon^2: \frac{d^2\theta_2(x)}{dx^2} - \theta_2(x) - 2\theta_0(x)\theta_1(x) = 0 \quad (35)$$

$$\theta_2(1) = 1, \quad \theta_2'(0) = 0$$

The solutions of eqs. (32)-(34) are:

$$\theta_0(x) = \frac{e^{-x}}{e^{-1} + e} + \frac{e^x}{e^{-1} + e} \quad (36)$$

$$\theta_1(x) = \frac{1}{3} \frac{e^x(-e^4 + 6e^2 - 1)}{e^{-1} + 2e^2e^{-1} + e^{-1}e^4 + e + 2ee^2 + ee^4} + \frac{1}{3} \frac{e^{-x}(-e^4 + 6e^2 - 1)}{e^{-1} + 2e^2e^{-1} + e^{-1}e^4 + e + 2ee^2 + ee^4} + \frac{e^{-2x+2}(1 - 6e^{2x} + e^{4x})e^2}{3 + 6e^2 + 3e^4} \quad (37)$$

$$\theta_2(x) = -\frac{1}{3} \frac{1}{(20e^6 + 15e^4 + 6e^2 + 15e^8 + 6e^{10} + e^{12} + 1)} (-18e^{2x+10}32e^{2x+8} - 34e^{-x+9} - 180e^{x+7}x + 60e^{-x+3}x - 60e^{x+9}x - 60e^{x+3}x + 60e^{-x+9}x + 180e^{-x+7}x + 180e^{-x+5}x - 180e^{x+5}x + 145e^{-x+5} + 80e^{2x+6} + 32e^{-2x+8} + 3e^{3x+3} - 154e^{-x+3} + 32e^{4+2x} - 154e^{-x+3} + 32e^{2x+4} - 154e^{x+3} + 48e^2 + 5e^{x+1} + 48e^{10} + 5e^{1-x} + 3e^{9+3x}x + 145e^{x+5} + 5e^{x+11} + 256e^{x+7} - 34e^{x+9} - 480e^6 + 9e^{-3x+5} - 192e^8 - 8e^{-2x+2} + 3e^{-3x+9} + 9e^{-3x+7} + 5e^{11-x} + 9e^{3x+7} - 8e^{-2x+10} + 256e^{-x+7} + 32e^{-2x+4} + 80e^{-2x+6} - 8e^{2x+2} + 9e^{3x+5} + 3e^{-3x+3}) \quad (38)$$

The approximation solution obtained by perturbation method will be:

$$\theta(x) = \theta_0(x) + \varepsilon\theta_1(x) + \varepsilon^2\theta_2(x) \quad (39)$$

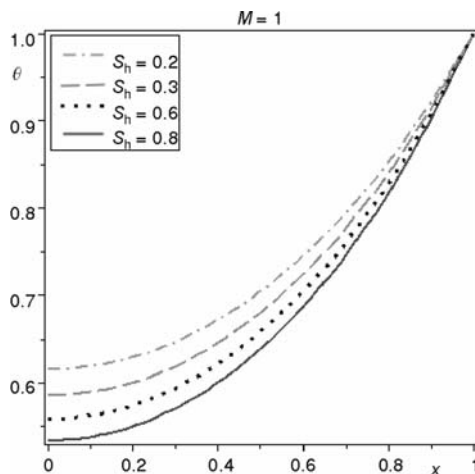


Figure 2. The distribution of axial non-dimensional temperature along the finite fin for different values of S_h with VIM

Results and discussion

An analytical solution for the temperature distributions in the fin by different method approximant was obtained. The results are compared with HPM, VIM, PM, and accurate numerical solution using BVP. Figures 3-5, depict the temperature distribution with the axial distance along the fin for the three methods. It is observed that the VIM approximant solution is more accurate than other methods. Comparing figs. 3-5, gives closer results to numerical solution. It is interesting to note that VIM is very close to the numerical results.

Figure 2, shows the variation of dimensionless temperature distribution with the axial distance along the fin when the value of S_h is varying and ε is constant. From fig. 2, we can see that the value of dimensionless temperature decreases

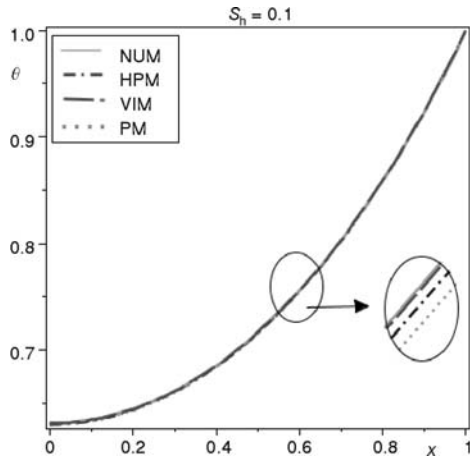


Figure 3. The comparison of the results of the methods at $M = 1$, and $S_h = 0.1$

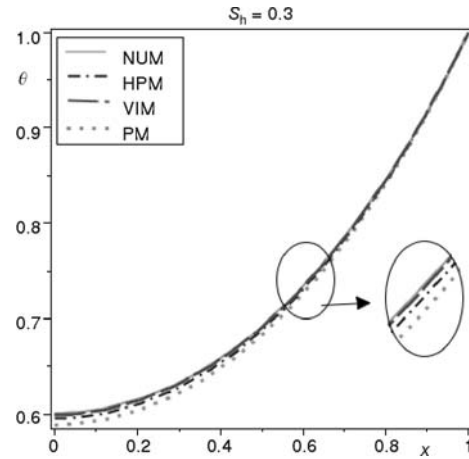


Figure 4. The comparison of the results of the methods at $M = 1$, and $S_h = 0.3$

along the fin Length. It should be noted that as the value of θ increases, the temperature decreases rapidly and the fin quickly reaches the surrounding temperature. As the value of S_h increases, the fins cool down rapidly.

Conclusion

The present work, the basic idea of the variational iteration method, homotopy perturbation method and perturbation method are introduced and then we have applied to methods to solve the temperature distribution of a porous fin because a second order non-linear ordinary differential equation has been derived as the governing equation for this problem. Result demonstrates the variational iteration method is simple and offers superior accuracy compared with the perturbation method and homotopy perturbation method. Also it is found that these method are powerful mathematical tools and that they can be applied to a large class of linear and non-linear problems arising in different fields of science and engineering specially some heat transfer equations.

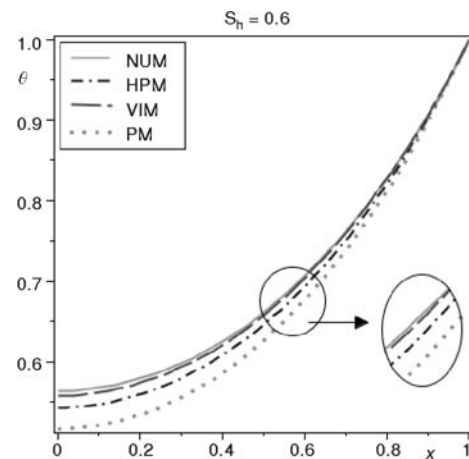


Figure 5. The comparison of the results of the methods at $M = 1$, and $S_h = 0.6$

Nomenclature

- A – cross-sectional area, [m²]
- C_p – specific heat
- Da – Darcy number, ($= k/t^2$)
- Gr – Grashof number ($= \beta g(T_w - T)L^3/\nu^2$)
- h – heat transfer coefficient, [Jm⁻²K⁻¹]
- K – thermal conductivity, [Wm⁻¹K⁻¹]

- K_r – thermal conductivity ratio, (k_{eff}/k_f)
- k – permeability of the porous fin, [m]
- M – convection parameter ($= hp/k_s A$)^{1/2}
- \dot{m} – mass flow rate, [kgs⁻¹]
- Nu – Nusselt number, (hL/k_f)
- Pr – Prandtl number, (ν/α)

| | | | |
|------------|--|-------------------|--|
| p | – fin perimeter, [m] | ε | – porosity |
| q | – heat flux, [Jm^{-2}] | σ | – Stephen-Boltzmann constant |
| Ra | – Rayleigh number, (Gr Pr) | θ | – dimensionless temperature |
| S_h | – porous parameter ($= \text{Da Ra}L^2/k_t^2$) | θ_b | – surface temperature parameter, ($= T_b - T_\infty$) |
| $T(x)$ | – temperature at any point, [K] | μ | – dynamic viscosity, [$\text{kgm}^{-1}\text{s}^{-1}$] |
| T_b | – temperature at fin base, [K] | ν | – kinematic viscosity, [m^2s^{-1}] |
| t | – thickness of the fin, [m] | σ | – Stefan-Boltzman constant, [Wm^2K^2] |
| $V_{w(x)}$ | – average velocity of the fluid passing through the fin at an point, [ms^{-1}] | ρ | – density of the fluid, [kgm^{-3}] |
| w | – width of the fin, [m] | Subscripts | |
| X | – dimensionless axial co-ordinate, (x/L) | b | – conditions at the fin base |
| x | – axial co-ordinate, [m] | eff | – porous properties |

Greek symbols

| | |
|----------|--|
| β | – coefficient of volumetric thermal expansion |
| Δ | – temperature difference |

| | |
|----------|----------------------|
| f | – fluid properties |
| s | – solid properties |
| ∞ | – ambient conditions |

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